

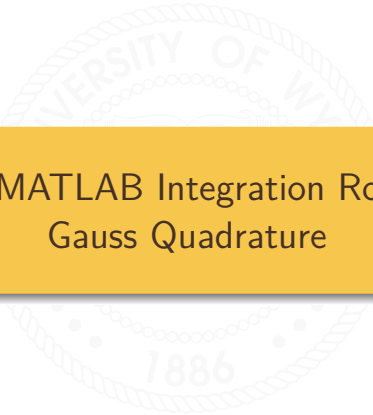
MATH 3341: Introduction to Scientific Computing Lab

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Lab 11: MATLAB Integration Routines & Gauss Quadrature



The background of the slide features a large, faint watermark of the University of Wyoming seal. The seal is circular with a rope-like border. Inside the border, the words "UNIVERSITY OF WYOMING" are at the top, "EQUALITY" is in the center, and "1886" is at the bottom. In the middle of the seal is an open book.

Built-in Integration Routines



polyint Integrate polynomial analytically

- `polyint(P,K)` returns a polynomial representing the integral of polynomial `P`, using a scalar constant of integration `K`.
- `polyint(P)` assumes a constant of integration `K=0`.



trapz Trapezoidal numerical integration

$Z = \text{trapz}(X,Y)$ computes the integral of Y with respect to X using the trapezoidal method. X and Y must be vectors of the same length, or X must be a column vector and Y an array whose first non-singleton dimension is $\text{length}(X)$. trapz operates along this dimension.

Let $X = [x_1, x_2, \dots, x_n]$, $Y = [y_1, y_2, \dots, y_n]$, then

$$Z = \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)(y_{i+1} + y_i)}{2} = \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)(y_{i+1} + y_i).$$



trapz Trapezoidal numerical integration

- $Z = \text{trapz}(Y)$ computes an approximation of the integral of Y via the trapezoidal method (with unit spacing). To compute the integral for spacing different from one, multiply Z by the spacing increment.
- $Z = \text{trapz}(X,Y,DIM)$ or $\text{trapz}(Y,DIM)$ integrates across dimension DIM of Y . The length of X must be the same as $\text{size}(Y,DIM)$.



cumtrapz Cumulative trapezoidal numerical integration

- $Z = \text{cumtrapz}(Y)$ computes an approximation of the cumulative integral of Y via the trapezoidal method (with unit spacing). To compute the integral for spacing different from one, multiply Z by the spacing increment.
- $Z = \text{cumtrapz}(X,Y)$ computes the cumulative integral of Y with respect to X using trapezoidal integration. X and Y must be vectors of the same length, or X must be a column vector and Y an array whose first non-singleton dimension is $\text{length}(X)$. cumtrapz operates across this dimension.
- $Z = \text{cumtrapz}(X,Y,\text{DIM})$ or $\text{cumtrapz}(Y,\text{DIM})$ integrates along dimension DIM of Y . The length of X must be the same as $\text{size}(Y,\text{DIM})$.



integral Numerically evaluate integral.

- $Q = \text{integral}(\text{FUN}, A, B)$ approximates the integral of function FUN from A to B using global adaptive quadrature and default error tolerances.
FUN must be a function handle. A and B can be $-\text{Inf}$ or Inf . If both are finite, they can be complex. If at least one is complex, integral approximates the path integral from A to B over a straight line path.
- $Q = \text{integral}(\text{FUN}, A, B, \text{PARAM1}, \text{VAL1}, \text{PARAM2}, \text{VAL2}, \dots)$ performs the integration with specified values of optional parameters.



integral2 Numerically evaluate double integral

- $Q = \text{integral2}(\text{FUN}, \text{XMIN}, \text{XMAX}, \text{YMIN}, \text{YMAX})$ approximates the integral of $\text{FUN}(X,Y)$ over the planar region $\text{XMIN} \leq X \leq \text{XMAX}$ and $\text{YMIN}(X) \leq Y \leq \text{YMAX}(X)$. FUN is a function handle, YMIN and YMAX may each be a scalar value or a function handle.
- $Q = \text{integral2}(\text{FUN}, \text{XMIN}, \text{XMAX}, \text{YMIN}, \text{YMAX}, \text{PARAM1}, \text{VAL1}, \text{PARAM2}, \text{VAL2})$ performs the integration as above with specified values of optional parameters



integral3 Numerically evaluate triple integral

- $Q = \text{integral3}(\text{FUN}, \text{XMIN}, \text{XMAX}, \text{YMIN}, \text{YMAX}, \text{ZMIN}, \text{ZMAX})$ approximates the integral of $\text{FUN}(X,Y,Z)$ over the region $\text{XMIN} \leq X \leq \text{XMAX}$, $\text{YMIN}(X) \leq Y \leq \text{YMAX}(X)$, and $\text{ZMIN}(X,Y) \leq Z \leq \text{ZMAX}(X,Y)$. FUN is a function handle, YMIN , YMAX , ZMIN , and ZMAX may each be a scalar value or a function handle.
- $Q = \text{integral3}(\text{FUN}, \text{XMIN}, \text{XMAX}, \text{YMIN}, \text{YMAX}, \text{ZMIN}, \text{ZMAX}, \text{PARAM1}, \text{VAL1}, \text{PARAM2}, \text{VAL2}, \dots)$ performs the integration as above with specified values of optional parameters





Gauss-Legendre Quadrature



Gauss-Legendre Quadrature on $[-1, 1]$

Integration of $f(x)$ on the interval $[-1, 1]$ using Gauss Quadrature is given by

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i),$$

where w_i and x_i are chosen so the integration rule is exact for the largest class of polynomials. $f(x)$ is well-approximated by polynomial on $[-1, 1]$, the associated orthogonal polynomials are *Legendre polynomial*, denoted by $P_n(x)$. With the n -th polynomial normalized to give $P_n(1) = 1$, the i -th Gauss node, x_i , is the i -th root of P_n and the weights are given by the formula (Abramowitz & Stegun 1972, p. 887):

$$w_i = \frac{2}{(1 - x_i^2)[P'_n(x_i)]^2}.$$



Gauss-Legendre Quadrature on $[a, b]$

To approximate the integral on the general interval $[a, b]$, we need to use the change of variables as follows:

$$\begin{aligned}\frac{t-a}{b-a} &= \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2} \implies t = \frac{b-a}{2}x + \frac{b+a}{2}, -1 \leq x \leq 1 \\ &\implies dt = \frac{b-a}{2}dx.\end{aligned}$$

So the Gauss Quadrature on a general interval $[a, b]$ is given by

$$\begin{aligned}\int_a^b f(t) dt &= \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2} dx \\ &\approx \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right) \frac{b-a}{2}.\end{aligned}$$



Gauss-Legendre Quadrature on $[a, b]$

Let

$$g(x) = f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2},$$

then

$$\int_a^b f(t) dt = \int_{-1}^1 g(x) dx \approx \sum_{i=1}^n w_i g(x_i).$$

