MATH 3341: Introduction to Scientific Computing Lab

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Lab 12: Romberg Integration

Romberg Integration

Composite Trapezoidal rule for approximating the integral of a function $f(x)$ on an interval $[a, b]$ using m subintervals is

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{m-1} f(x_j)\right] - \frac{b-a}{12}h^2 f''(\xi),$$

where $a \le \xi \le b$ and $h = (b-a)/m$ and $x_j = a + jh$ for each $j = 0, 1, \ldots, m$.

Finding approximation for $m_1 = 1, m_2 = 2, m_3 = 4, \ldots, m_n = 2^{n-1}$ for $n \in \mathbb{N}$. The corresponding step size $h_k$for each m_k is then given by $h_k = (b-a)/m_k = (b-a)/2^{k-1}$. The trapezoidal rule then becomes

$$\int_a^b f(x)\,dx = \frac{h_k}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{2^{k-1}-1} f(a + jh_k)\right] - \frac{b-a}{12}h^2 f''(\xi_k),$$

where $\xi_k \in [a, b]$.

Here we'll use the notation $R_{k,1}$ to denote the portion used for the trapezoidal approximation. In other words,

$$
\begin{aligned}
R_{1,1} &= \frac{h_1}{2}[f(a)+f(b)] = \frac{b-a}{2}[f(a)+f(b)],\\
R_{2,1} &= \frac{h_2}{2}[f(a)+f(b)+2f(a+h_2)]\\
&= \frac{1}{2}\frac{b-a}{2}\left[f(a)+f(b)+2f\left(a+\frac{b-a}{2}\right)\right]\\
&= \frac{1}{2}[R_{1,1}+h_1 f(a+h_2)],\\
R_{3,1} &= \frac{1}{2}\{R_{2,1}+h_2[f(a+h_3)+f(a+3h_3)]\}.
\end{aligned}
$$

This leads to the Trapezoidal rule in the general form

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{j=1}^{2^{k-2}} f(a+(2j-1)h_k)\right] \quad \text{for } k = 2, 3, \ldots, n.$$

This method converges very slowly on its own. A technique called Richardson's Extrapolation is applied to speed convergence. Essentially, this performs a method of averaging previously calculated entries to obtain the next entry in the table. This is given in general form

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}.$$

This method will give us the following entries of R in a tabular format. The number of rows is determined by the value that we desire.

$$\begin{array}{cccccc}
R_{1,1} & & & & & \\
R_{2,1} & R_{2,2} & & & & \\
R_{3,1} & R_{3,2} & R_{3,3} & & & \\
R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
R_{n,1} & R_{n,2} & R_{n,3} & R_{n,4} & \cdots & R_{n,n}
\end{array}$$

Algorithm

Algorithm 1: Romberg Integration

Function romberg(f, a, b, n)**:**

  $h \leftarrow b - a$;

  $R_{1,1} \leftarrow [f(a) + f(b)] \cdot h/2$;

  for $k \leftarrow 2$ **to** n **do**

    $$R_{k,1} \leftarrow \frac{1}{2}\left[R_{k-1,1} + h\sum_{j=1}^{2^{k-2}} f(a + (2j-1)\cdot h/2)\right];$$

    for $j \leftarrow 2$ **to** k **do**

      $$R_{k,j} \leftarrow R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1};$$

    end

    $h \leftarrow h/2$;

  end

  return $[R_{1,1}, R_{2,2}, R_{3,3}, \ldots, R_{n,n}]$;

end