Full Name: ______________________ **W-Number:** ________________

MATH 2205: Calculus II – Sample Exam 2 Solution

Summer 2019 – Friday, June 21, 2019

Instructions:

- Show all your work and use the space provided on the exam. Correct mathematical notation is required and all partial credit is at discretion of the grader.
- Write neatly and make sure your work is organized.
- Make certain that you have written your Full Name and W-Number in the spaces provided at the top of the exam. Failure to do so may result in a loss of points.
- No aids beyond a scientific, non-graphing calculator are allowed. This means no notes, no cell phones, etc., are permitted during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 12 pages. Please check to see that your copy has all the pages.

For Instructor Use Only

Question	1	2	3	4	5	6	7	8	9	Total
Points	15	20	10	10	10	10	10	15	10	100
Mark										

Formulae you may find useful:

- $\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$
- $\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.
- $\int x^{-1} \, dx = \ln|x| + C.$
- $\int e^x \, dx = e^x + C.$
- $\int \sin x \, dx = -\cos x + C.$
- $\int \cos x \, dx = \sin x + C.$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C.$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C.$
- $\int \sec x \tan x \, dx = \sec x + C.$
- $\int \sec^2 x \, dx = \tan x + C.$
- $\cos^2 \theta = \frac{1+\cos 2\theta}{2}.$
- $\sin^2 \theta = \frac{1-\cos 2\theta}{2}.$
- $\sin^2 \theta + \cos^2 \theta = 1.$
- $\sin 2\theta = 2 \sin\theta \cos\theta.$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta.$
- Midpoint Rule: $M(n) = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) \Delta x.$
- Trapezoid Rule: $T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right] \Delta x.$
- Simpson's Rule: $S(n) = \sum_{k=0}^{n/2-1} \left[f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right] \frac{\Delta x}{3}.$

1. (15 points) Circle TRUE if the statement is true or FALSE if it is not, and justify your choice briefly.

   (a) TRUE or **FALSE**: The partial fraction decomposition of $\frac{2x^3-4x^2+x+3}{(x^2+1)(x-2)^2}$ is $\frac{1}{x-2}+\frac{x+1}{x^2+1}$.

   Explanation: $\frac{2x^3-4x^2+x+3}{(x^2+1)(x-2)^2} = \frac{1}{x-2}+\frac{1}{(x-2)^2}+\frac{x+1}{x^2+1} \neq \frac{1}{x-2}+\frac{x+1}{x^2+1}$.

   Or $\frac{1}{x-2}+\frac{x+1}{x^2+1} = \frac{2x^2-x-1}{(x^2+1)(x-2)} \neq \frac{2x^3-4x^2+x+3}{(x^2+1)(x-2)^2}$.

   (b) **TRUE** or FALSE: Use integration by parts, one can show that $\int xg'(x)\,dx = xg(x) - \int g(x)\,dx$.

   Explanation: By definition.

   (c) **TRUE** or FALSE: The function $y = e^{2t} - e^{-2t}$ is a solution of the differential equaiton $y'' - 4y = 0$.

   Explanation: Since $y' = 2e^{2t} + 2e^{-2t} \implies y'' = 4e^{2t} - 4e^{-2t} = 4(e^{2t} - e^{-2t}) = 4y \implies y'' - 4y = 0$.

   (d) TRUE or **FALSE**: The differential equation $y''(x) + (\ln x \cdot e^{e^x})y'(x) = y(x) + \cos x \sin e^x$ is NOT linear.

   Explanation: It is linear because it follows the form for second-order linear differential equation $y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$, where $p(x) = (\ln x \cdot e^{e^x})$, $q(x) = -1$, and $f(x) = \cos x \sin e^x$.

   (e) TRUE or **FALSE**: If $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty}(-1)a_k$ diverges.

   Explanation: By the property of Sigma notation, we have $\sum_{k=1}^{\infty}(-1)a_k = (-1)\sum_{k=1}^{\infty} a_k$.

   It also converges given that $\sum_{k=1}^{\infty} a_k$ converges.

2. (20 points) Evaluate the following integrals and sums.

(a) $\displaystyle\int x^2(\ln x)^2\,dx$.

SOLUTION. It is integrating the product of x^p and $(\ln x)^q$, we pick $u = (\ln x)^q$, then apply integration by parts for this indefinite integral.

$$\begin{aligned}
\int x^2(\ln x)^2\,dx &= \int (\ln x)^2\,d\left(\frac{1}{3}x^3\right) && [u=(\ln x)^2, dv = x^2\,dx \implies v = \frac{1}{3}x^3] \\
&= \frac{1}{3}x^3(\ln x)^2 - \int \frac{1}{3}x^3\,d(\ln x)^2 && [\int u\,dv = uv - \int v\,du] \\
&= \frac{1}{3}x^3(\ln x)^2 - \int \frac{1}{3}x^3 \cdot 2\ln x \cdot \frac{1}{x}\,dx \\
&= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2 \ln x\,dx \\
&= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int \ln x\,d\left(\frac{1}{3}x^3\right) && [u = \ln x, dv = x^2\,dx \implies v = \frac{1}{3}x^3] \\
&= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left(\frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3\,d\ln x\right) && [\int u\,dv = uv - \int v\,du] \\
&= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left(\frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx\right) \\
&= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C.
\end{aligned}$$

□

(b) $\displaystyle\int_0^{\pi/2} \sin^3 x\cos^4 x\,dx$.

SOLUTION. Since the power of $\sin x$ is odd, then we split off $\sin x$, then use the identity $\sin^2 x + \cos^2 x = 1$ and then use change of variables.

$$\begin{aligned}
\int_0^{\pi/2} \sin^3 x\cos^4 x\,dx &= \int_0^{\pi/2} \sin x(1-\cos^2 x)\cos^4 x\,dx && [\sin^2 x = 1 - \cos^2 x] \\
&= -\int_{u(0)=1}^{u(\pi/2)=0} (1-u^2)u^4\,du && [u = \cos x, du = -\sin x dx] \\
&= -\left(\int_1^0 u^4\,du - \int_1^0 u^6\,du\right) \\
&= -\left(\frac{u^5}{5}\Bigg|_1^0 - \frac{u^7}{7}\Bigg|_1^0\right) \\
&= -\left[\left(0-\frac{1}{5}\right) - \left(0 - \frac{1}{7}\right)\right] \\
&= \frac{2}{35}.
\end{aligned}$$

□

(c) $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{5}{3}\right)^{-k}$.

SOLUTION.

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(-\frac{5}{3}\right)^{-k} &= \sum_{k=1}^{\infty}\left[\left(-\frac{5}{3}\right)^{-1}\right]^{k} && [a^{bc}=(a^b)^c]\\
&= \sum_{k=1}^{\infty}\left(-\frac{3}{5}\right)^{k} && [a^{-1}=\frac{1}{a}]\\
&= \frac{-3/5}{1-(-3/5)} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}]\\
&= \frac{-3/5}{8/5}\\
&= -\frac{3}{8}.
\end{aligned}$$

□

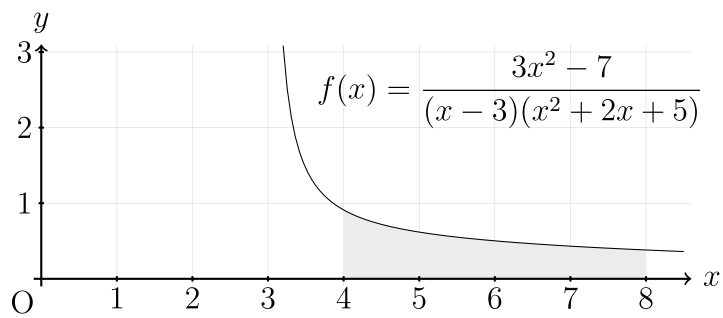
(d) $\displaystyle\frac{1}{16}+\frac{3}{64}+\frac{9}{256}+\frac{27}{1024}+\cdots$.

SOLUTION.

$$\begin{aligned}
\frac{1}{16}+\frac{3}{64}+\frac{9}{256}+\frac{27}{1024}+\cdots &= \sum_{k=1}^{\infty}\frac{3^{k-1}}{4^{k+1}}\\
&= \sum_{k=1}^{\infty}\frac{3^k}{4^k}\frac{3^{-1}}{4^1}\\
&= \frac{1}{12}\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k}\\
&= \frac{1}{12}\frac{3/4}{1-3/4} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}]\\
&= \frac{1}{12}\times 3\\
&= \frac{1}{4}.
\end{aligned}$$

□

3. (10 points) Compute the area of region R bounded by $f(x) = \dfrac{3x^2-7}{(x-3)(x^2+2x+5)}$, x-axis, $x=4$, $x=8$. Indicate (by shading) the region R in the graph.



Solution. The area of the region R is

$$A = \int_a^b f(x)\,dx = \int_4^8 \frac{3x^2-7}{(x-3)(x^2+2x+5)}\,dx.$$

Then we apply the partial fraction decomposition to the integrand,

$$\frac{3x^2-7}{(x-3)(x^2+2x+5)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2x+5}.$$

Multiplying both sides by $(x-3)(x^2+2x+5)$ gives

$$3x^2-7 = A(x^2+2x+5)+(Bx+C)(x-3) = (A+B)x^2+(2A-3B+C)x+(5A-3C).$$

Equating like powers of x, we have the following linear equations,

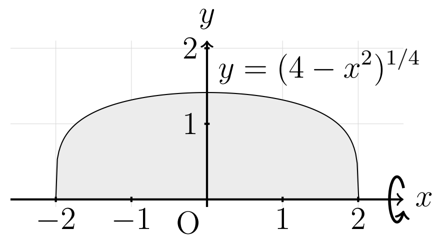
$$\begin{cases} A+B=3 \\ 2A-3B+C=0 \\ 5A-3C=-7 \end{cases} \implies \begin{cases} A=1 \\ B=2 \\ C=4 \end{cases}$$

Then definite integral becomes

$$
\begin{aligned}
A &= \int_4^8 \frac{3x^2-7}{(x-3)(x^2+2x+5)}\,dx \\
&= \int_4^8 \frac{1}{x-3} + \frac{2x+4}{x^2+2x+5}\,dx \\
&= \int_{u(4)=4-3=1}^{u(8)=8-3=5} \frac{1}{u}\,du + \int_4^8 \frac{2x+2}{x^2+2x+5}\,dx + \int_4^8 \frac{2}{x^2+2x+5}\,dx \\
&= \ln|u|\Big|_1^5 + \int_{v(4)=29}^{v(8)=85} \frac{1}{v}\,dv + 2\int_4^8 \frac{1}{(x^2+2x+1)+4}\,dx \\
&= \ln 5 + \ln|v|\Big|_{29}^{85} + 2\int_4^8 \frac{1}{4+(x+1)^2}\,dx \\
&= \ln 5 + \ln 85 - \ln 29 + 2\int_4^8 \frac{1}{4[1+\frac{1}{4}(x+1)^2]}\,dx \\
&= \ln(5\times 85) - \ln 29 + 2\int_4^8 \frac{1}{4}\frac{1}{1+(x/2+1/2)^2}\,dx \\
&= \ln\frac{425}{29} + \int_{w(4)=5/2}^{w(8)=9/2} \frac{1}{1+w^2}\,dw \\
&= \ln\frac{425}{29} + \arctan w\Big|_{5/2}^{9/2} \\
&= \ln\frac{425}{29} + \arctan\frac{9}{2} - \arctan\frac{5}{2}.
\end{aligned}
$$

□

4. (10 points) Let R be the region bounded by $y = (4 - x^2)^{1/4}$ and x-axis. Indicate (by shading) the region R in the graph. Determine the volume of the solid of revolution obtained when R is revolved about the x-axis.

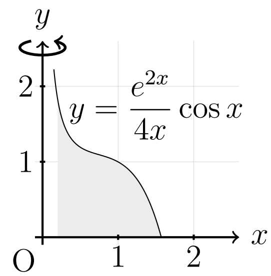


SOLUTION. Disk method.

$$\begin{aligned}
V &= \int_a^b A(x)\,dx \\
&= \int_{-2}^{2} \pi f(x)^2\,dx \\
&= \pi \int_{-2}^{2} (4 - x^2)^{1/2}\,dx \\
&= \pi \int_{\arcsin -1 = -\pi/2}^{\arcsin 1 = \pi/2} (4 - 4\sin^2\theta)^{1/2} \cos\theta\,d\theta \quad [x = 2\sin\theta, dx = 2\cos\theta\,d\theta, \theta = \arcsin\frac{x}{2}] \\
&= \pi \int_{-\pi/2}^{\pi/2} 2\cos\theta \cdot 2\cos\theta\,d\theta \\
&= \pi \int_{-\pi/2}^{\pi/2} 4\cos^2\theta\,d\theta \\
&= 2\pi \int_{-\pi/2}^{\pi/2} 1 + \cos 2\theta\,d\theta \\
&= 2\pi \left(\int_{-\pi/2}^{\pi/2} 1\,d\theta + \int_{-\pi/2}^{\pi/2} \cos 2\theta\,d\theta \right) \\
&= 2\pi \left(\theta \Big|_{-\pi/2}^{\pi/2} + \frac{1}{2} \int_{u(-\pi/2) = -\pi}^{u(\pi/2) = \pi} \cos u\,du \right) \\
&= 2\pi \left(\pi + \frac{1}{2} \sin u \Big|_{-\pi}^{\pi} \right) \\
&= 2\pi^2.
\end{aligned}$$

□

5. (10 points) Let R be the region bounded by $y = \dfrac{e^{2x}}{4x}\cos x$ and x-axis on $[1/5, \pi/2]$. Indicate (by shading) the region R in the graph. Determine the volume of the solid of revolution obtained when R is revolved about the y-axis.



SOLUTION. Shell method.

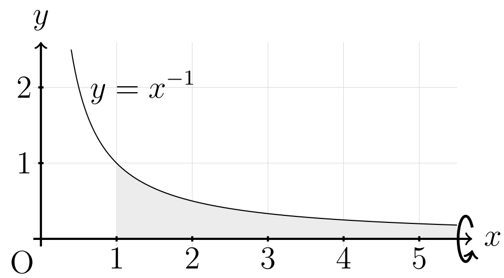
$$
\begin{aligned}
V = \int_a^b 2\pi x f(x)\,dx &= \int_{1/5}^{\pi/2} 2\pi x \frac{e^{2x}}{4x}\cos x\,dx \\
&= \frac{\pi}{2}\int_{1/5}^{\pi/2} e^{2x}\,d\sin x \\
&= \frac{\pi}{2}\left(e^{2x}\sin x\Big|_{1/5}^{\pi/2} - \int_{1/5}^{\pi/2} 2\sin x e^{2x}\,dx \right) \\
&= \frac{\pi}{2}\left(e^{2x}\sin x\Big|_{1/5}^{\pi/2} + 2\int_{1/5}^{\pi/2} e^{2x}\,d\cos x \right) \\
&= \frac{\pi}{2}\left[e^{2x}\sin x\Big|_{1/5}^{\pi/2} + 2\left(e^{2x}\cos x\Big|_{1/5}^{\pi/2} - 2\int_{1/5}^{\pi/2} e^{2x}\cos x\,dx \right)\right] \\
&= \frac{\pi}{2} e^{2x}(\sin x + 2\cos x)\Big|_{1/5}^{\pi/2} - 2\pi\int_{1/5}^{\pi/2} e^{2x}\cos x\,dx.
\end{aligned}
$$

Then solve for $\frac{\pi}{2}\int_{1/5}^{\pi/2} e^{2x}\cos x\,dx$, we have the volume of the solid as follows

$$
\begin{aligned}
\frac{\pi}{2}\int_{1/5}^{\pi/2} e^{2x}\cos x\,dx &= \frac{1}{5}\frac{\pi}{2} e^{2x}(\sin x + 2\cos x)\Big|_{1/5}^{\pi/2} \\
&= \frac{\pi}{10}\left[e^{\pi}\left(\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2}\right) - e^{2/5}\left(\sin\frac{1}{5} + 2\cos\frac{1}{5})\right]\right. \\
&= \frac{\pi}{10}\left[e^{\pi} - e^{2/5}\left(\sin\frac{1}{5} + 2\cos\frac{1}{5}\right)\right].
\end{aligned}
$$

□

6. (10 points) Let R be the region bounded by the graph of $y = x^{-1}$ and the x-axis, for $x \geq 1$. Indicate (by shading) the region R in the graph. What is the surface area of the solid generated when R is revolved about the x-axis?



Solution. Given that $f(x) = x^{-1}$, $f'(x) = -x^{-2} \implies f'(x)^2 = x^{-4}$. The surface area of the region is

$$\begin{aligned}
A &= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \\
&= \int_1^\infty 2\pi x^{-1}\sqrt{1+x^{-4}}\,dx \\
&= 2\pi \int_1^\infty x^{-1}\sqrt{x^{-4}(x^4+1)}\,dx \\
&= 2\pi \int_1^\infty x^{-1}(x^{-4})^{1/2}[(x^4+1)]^{1/2}\,dx \\
&= 2\pi \int_1^\infty x^{-3}[(x^4+1)]^{1/2}\,dx \\
&> 2\pi \int_1^\infty x^{-3}(x^4)^{1/2}\,dx \\
&= 2\pi \int_1^\infty x^{-3}x^2\,dx \\
&= 2\pi \int_1^\infty x^{-1}\,dx \\
&= 2\pi \lim_{b\to\infty} \int_1^b x^{-1}\,dx \\
&= 2\pi \lim_{b\to\infty} \ln x \Big|_1^b \\
&= 2\pi \lim_{b\to\infty} \ln b \\
&= \infty.
\end{aligned}$$

Therefore, the surface area of the solid diverges. □

7. (10 points) Write $1.0\overline{95} = 1.095959595\ldots$ as a geometric series and express its value as fraciton.

Solution.

$$\begin{aligned}
1.0\overline{95} &= 1.095959595\ldots \\
&= 1 + 0.095 + 0.00095 + 0.0000095 + \cdots \qquad [a = 0.095, r = \frac{1}{100}, |r| < 1] \\
&= 1 + \sum_{k=0}^{\infty} 0.095 \cdot \left(\frac{1}{100}\right)^k \\
&= 1 + \frac{0.095}{1 - 1/100} \qquad [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}] \\
&= 1 + \frac{95/1000}{99/100} \\
&= 1 + \frac{95}{990} \\
&= 1 + \frac{19}{198} \\
&= \frac{217}{198}.
\end{aligned}$$

□

8. (15 points) A community of hares on an island has a population of 70 when observations begin (at $t = 0$). The population is modeled by the initial value problem shown below.

$$\frac{dP}{dt} = 0.07P\left(1 - \frac{P}{280}\right), \quad P(0) = 70.$$

(a) Find the solution of the initial value problem, for $t \geq 0$.
(b) What is the steady-state population?

Solution.

(a) Observe that this differential equation is separable, that is,

$$\begin{aligned}\frac{dP}{dt} &= 0.07P\left(1 - \frac{P}{280}\right)\\ &= \frac{7P}{100}\frac{280 - P}{280}\\ &= \frac{(280 - P)P}{4000}\\ &= f(P).\end{aligned}$$

Then dividing both sides by $f(P)$ and integrating with respect to t gives

$$\begin{aligned}\int \frac{1}{f(P)}\frac{dP}{dt}\,dt &= \int 1\,dt\\ \int \frac{1}{f(P)}\,dP &= \int 1\,dt\\ \int \frac{4000}{(280 - P)P}\,dP &= \int 1\,dt\\ \int \frac{100}{7P} + \frac{100}{7(280 - P)}\,dP &= \int 1\,dt \quad \text{[Partial Fractions Decomposition]}\\ \frac{100}{7}\int \frac{1}{P}\,dP + \frac{100}{7}\int \frac{1}{280 - P}\,dP &= \int 1\,dt\\ \frac{100}{7}\ln|P| - \frac{100}{7}\ln|280 - P| &= t + C_1\\ \frac{100}{7}\ln\left|\frac{P}{280 - P}\right| &= t + C_1\end{aligned}$$

Then solve for P, we have

$$\begin{aligned}
\ln\left|\frac{P}{280-P}\right| &= \frac{7}{100}t + \frac{7}{100}C_1 \\
\frac{P}{280-P} &= e^{\frac{7}{100}t+\frac{7}{100}C_1} \\
\frac{P}{280-P} &= C_2 e^{\frac{7}{100}t} \\
\frac{280-P}{P} &= \frac{1}{C_2}e^{-\frac{7}{100}t} \\
\frac{280}{P} &= \frac{1}{C_2}e^{-\frac{7}{100}t} + 1 \\
P &= \frac{280}{e^{-\frac{7}{100}t}/C_2 + 1}.
\end{aligned}$$

By the initial condition, we have

$$P(0) = \frac{280}{1/C_2 + 1} = 70 \implies 1/C_2 + 1 = 4 \implies C_2 = 1/3.$$

Therefore the solution of the intial value problem is

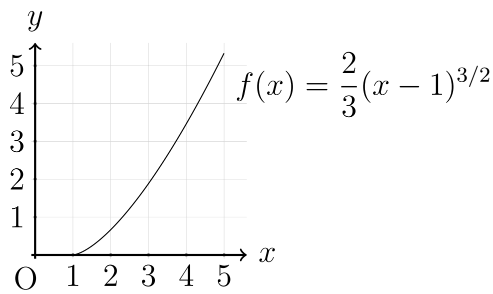
$$P = \frac{280}{3e^{-7t/100} + 1}.$$

(b) The steady-state population is obtained when $t \to \infty$, hence we have

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{280}{3e^{-7t/100} + 1} = \frac{280}{\lim_{t\to\infty} 3e^{-7t/100} + 1} = 280.$$

□

9. (Bonus, 10 points) Approximate the arc length of $f(x) = \dfrac{2}{3}(x-1)^{3/2}$ on the interval $[1, 5]$ using Simpson's Rule with $n = 4$ subintervals.



Solution. Given that $f(x) = \frac{2}{3}(x-1)^{3/2}$, we have $f'(x) = (x-1)^{1/2} \implies f'(x)^2 = (x-1)$. Then the arc length is

$$
\begin{aligned}
L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\
&= \int_1^5 \sqrt{1 + (x-1)}\, dx \\
&= \int_1^5 \sqrt{x}\, dx.
\end{aligned}
$$

Then we approximate the definite integral using Simpson's Rule as follows. First,

$$\Delta x = \frac{b-a}{n} = \frac{4}{4} = 1, x_k = x_0 + k\Delta x = 1 + k.$$

Let $g(x) = \sqrt{x}$. Then we have

$$
\begin{aligned}
\int_1^5 \sqrt{x}\, dx = \int_1^5 g(x)\, dx &\approx \sum_{k=0}^{n/2-1} [g(x_{2k}) + 4g(x_{2k+1}) + g(x_{2k+2})] \frac{\Delta x}{3} \\
&= \frac{\Delta x}{3} \sum_{k=0}^{n/2-1} [g(x_{2k}) + 4g(x_{2k+1}) + g(x_{2k+2})] \\
&= \frac{1}{3} \sum_{k=0}^{1} [g(1+2k) + 4g(1+2k+1) + g(1+2k+2)] \\
&= \frac{1}{3} [(\sqrt{1} + 4\sqrt{2} + \sqrt{3}) + (\sqrt{3} + 4\sqrt{4} + \sqrt{5})] \\
&= \frac{1}{3} (1 + 4\sqrt{2} + 2\sqrt{3} + 4 \cdot 2 + \sqrt{5}) \\
&\approx 6.78567.
\end{aligned}
$$

□