Full Name: ______________________ **W-Number:** ________________

MATH 2205: Calculus II – Midterm Exam 2 Solution

Summer 2019 – Friday, June 21, 2019

Instructions:

- Show all your work and use the space provided on the exam. Correct mathematical notation is required and all partial credit is at discretion of the grader.
- Write neatly and make sure your work is organized.
- Make certain that you have written your Full Name and W-Number in the spaces provided at the top of the exam. Failure to do so may result in a loss of points.
- No aids beyond a scientific, non-graphing calculator are allowed. This means no notes, no cell phones, etc., are permitted during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 10 pages. Please check to see that your copy has all the pages.

For Instructor Use Only

Question	1	2	3	4	5	6	7	8	9	Total
Points	15	20	10	10	10	10	10	15	10	100
Mark										

Formulae you may find useful:

- $\displaystyle\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$
- $\displaystyle\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.
- $\displaystyle\int x^{-1} \, dx = \ln|x| + C.$
- $\displaystyle\int e^x \, dx = e^x + C.$
- $\displaystyle\int \sin x \, dx = -\cos x + C.$
- $\displaystyle\int \cos x \, dx = \sin x + C.$
- $\displaystyle\int \frac{1}{1+x^2} \, dx = \arctan x + C.$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C.$
- $\displaystyle\int \sec x \tan x \, dx = \sec x + C.$
- $\displaystyle\int \sec^2 x \, dx = \tan x + C.$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}.$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}.$
- $\sin^2\theta + \cos^2\theta = 1.$
- $\sin 2\theta = 2\sin\theta\cos\theta.$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta.$
- Midpoint Rule: $\displaystyle M(n) = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x.$
- Trapezoid Rule: $\displaystyle T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x.$
- Simpson's Rule: $\displaystyle S(n) = \sum_{k=0}^{n/2-1} \left[f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right]\frac{\Delta x}{3}.$

1. (15 points) Circle TRUE if the statement is true or FALSE if it is not, and justify your choice briefly.

   (a) TRUE or $\boxed{\text{FALSE}}$: The partial fractions decomposition of $\dfrac{2x}{x^2+1}$ is $\dfrac{1}{x-1} + \dfrac{1}{x+1}$.
   Explanation: $\dfrac{1}{x-1} + \dfrac{1}{x+1} = \dfrac{2x}{(x-1)(x+1)} = \dfrac{2x}{x^2-1} \neq \dfrac{2x}{x^2+1}$.

   (b) TRUE or $\boxed{\text{FALSE}}$: $\displaystyle\int \cos^2 x\,dx = \left(\int \cos x\,dx\right)\left(\int \cos x\,dx\right)$.
   Explanation:
   $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}t + \frac{1}{4}\sin 2x + C.$$
   However, $\displaystyle\int \cos x\,dx = \sin x + C$. Then
   $$(\cos x\,dx)^2 = (\sin x + C_1)(\sin x + C_2) \neq \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

   (c) $\boxed{\text{TRUE}}$ or FALSE: The function $y = \cos 2t + \sin 2t$ is a solution of the differential equaiton $y'' + 4y = 0$.
   Explanation: Since $y' = -2\sin 2t + 2\cos 2t \implies y'' = -4\cos 2t - 4\sin 2t = -4(\cos 2t + \sin 2t) = -4y \implies y'' + 4y = 0$.

   (d) $\boxed{\text{TRUE}}$ or FALSE: The differential equation
   $$x^x y''(x) + (\ln x \cdot e^{e^x})y'(x) = \frac{y(x)}{x} + \cos x \sin e^x$$
   is linear.
   Explanation: It is linear because it follows the form for second-order linear differential equation $y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$, where $p(x) = \dfrac{\ln x \cdot e^{e^x}}{x^x}$, $q(x) = -\dfrac{1}{x^{x+1}}$, and $f(x) = \dfrac{\cos x \sin e^x}{x^x}$.

   (e) TRUE or $\boxed{\text{FALSE}}$: If $a_k > 0$ and $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then $\displaystyle\sum_{k=1}^{\infty}(-621^{2019})a_k$ diverges.
   Explanation: By the property of Sigma notation, we have $\displaystyle\sum_{k=1}^{\infty}(-621^{2019})a_k = (-621^{2019})\sum_{k=1}^{\infty} a_k$. It also converges given that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

2. (20 points) Evaluate the following integrals and sums.

(a) $\displaystyle\int \ln x \, dx.$

SOLUTION. It is integrating the product of x^p and $(\ln x)^q$, we pick $u = (\ln x)^q$, then apply integration by parts for this indefinite integral.

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \frac{1}{x} \, dx \qquad \left[u = \ln x, dv = dx \implies v = x, \int u \, dv = uv - \int v \, du\right]\\ &= x \ln x - \int 1 \, dx\\ &= x \ln x - x + C.\end{aligned}$$

□

(b) $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^3 x \, dx.$

SOLUTION. Since the powers of $\cos x$ and $\sin x$ are odd, then we split off $\cos x$, then use the identity $\sin^2 x + \cos^2 x = 1$ and then use change of variables.

$$\begin{aligned}\int_0^{\pi/2} \sin^7 x \cos^3 x \, dx &= \int_0^{\pi/2} \sin^7 x (1 - \sin^2 x) \cos x \, dx \qquad [\cos^2 x = 1 - \sin^2 x]\\ &= \int_{u(0)=0}^{u(\pi/2)=1} u^7 (1 - u^2) \, du \qquad [u = \sin x, du = \cos x \, dx]\\ &= \left(\int_0^1 u^7 \, du - \int_0^1 u^9 \, du\right)\\ &= \left(\frac{u^8}{8}\bigg|_0^1 - \frac{u^{10}}{10}\bigg|_0^1\right)\\ &= \left[\left(\frac{1}{8} - 0\right) - \left(\frac{1}{10} - 0\right)\right]\\ &= \frac{1}{40}.\end{aligned}$$

□

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{28}{9}\left(-\frac{4}{3}\right)^{-k}$.

SOLUTION.

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{28}{9}\left(-\frac{4}{3}\right)^{-k} &= \sum_{k=2}^{\infty} \frac{28}{9}\left[\left(-\frac{4}{3}\right)^{-1}\right]^{k} && [a^{bc} = (a^b)^c] \\
&= \sum_{k=2}^{\infty} \frac{28}{9}\left(-\frac{3}{4}\right)^{k} && [a^{-1} = \frac{1}{a}] \\
&= \frac{\frac{28}{9}\left(-\frac{3}{4}\right)^2}{1-(-3/4)} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}, a = \frac{28}{9}, r = -\frac{3}{4}, m = 2] \\
&= \frac{7/4}{7/4} \\
&= 1.
\end{aligned}$$

□

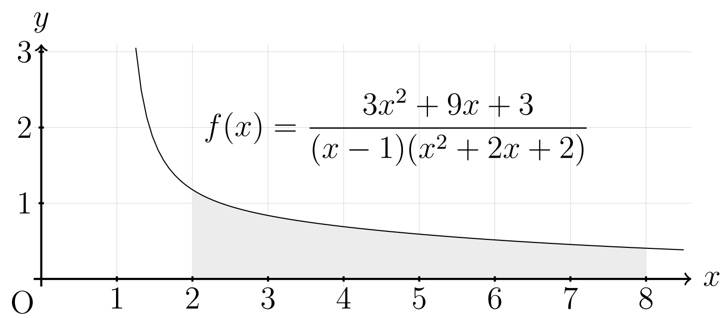
(d) $\dfrac{5}{4} + \dfrac{10}{12} + \dfrac{20}{36} + \dfrac{40}{108} + \cdots$.

SOLUTION.

$$\begin{aligned}
\frac{5}{4} + \frac{10}{12} + \frac{20}{36} + \frac{40}{108} + \cdots &= \sum_{k=0}^{\infty} \frac{5}{4}\left(\frac{2}{3}\right)^{k} \\
&= \frac{5/4}{1-2/3} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}, a = \frac{5}{4}, r = \frac{2}{3}, m = 0] \\
&= \frac{5}{4} \times 3 \\
&= \frac{15}{4}.
\end{aligned}$$

□

3. (10 points) Compute the area of region R bounded by $f(x) = \dfrac{3x^2+9x+3}{(x-1)(x^2+2x+2)}$, x-axis, $x=2$, $x=8$. Indicate (by shading) the region R in the graph.



Solution. The area of the region R is

$$A = \int_a^b f(x)\,dx = \int_2^8 \frac{3x^2+9x+3}{(x-1)(x^2+2x+2)}\,dx.$$

Then we apply the partial fraction decomposition to the integrand,

$$\frac{3x^2+9x+3}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}.$$

Multiplying both sides by $(x-1)(x^2+2x+2)$ gives

$$3x^2+9x+3 = A(x^2+2x+2)+(Bx+C)(x-1) = (A+B)x^2+(2A-B+C)x+(2A-C).$$

Equating like powers of x, we have the following linear equations,

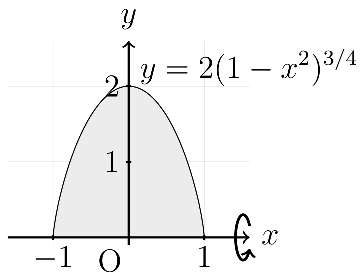
$$\begin{cases} A+B=3 \\ 2A-B+C=9 \\ 2A-C=3 \end{cases} \implies \begin{cases} A=3 \\ B=0 \\ C=3 \end{cases}$$

Then definite integral becomes

$$\begin{aligned} A = \int_2^8 \frac{3x^2+9x+3}{(x-1)(x^2+2x+2)}\,dx &= \int_2^8 \frac{3}{x-1} + \frac{3}{x^2+2x+2}\,dx \\ &= 3\int_2^8 \frac{1}{x-1}\,dx + 3\int_2^8 \frac{1}{(x^2+2x+1)+1}\,dx \\ &= 3\ln|x-1|\Big|_2^8 + 3\int_2^8 \frac{1}{(x+1)^2+1}\,dx \\ &= 3\ln|7| + 3\int_{u(2)=3}^{u(8)=9} \frac{1}{u^2+1}\,du \quad [u=x+1, du=dx] \\ &= 3\ln 7 + 3\arctan u\Big|_3^9 \\ &= 3\ln 7 + 3\arctan 9 - 3\arctan 3. \end{aligned}$$

$\square$

4. (10 points) Let R be the region bounded by $y = 2(1-x^2)^{3/4}$ and x-axis. Indicate (by shading) the region R in the graph. Determine the volume of the solid of revolution obtained when R is revolved about the x-axis.

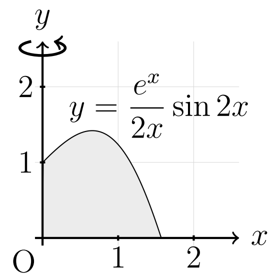


SOLUTION. Disk method.

$$
\begin{aligned}
V &= \int_a^b A(x)\,dx \\
&= \int_{-1}^{1} \pi f(x)^2\,dx \\
&= \int_{-1}^{1} \pi[2(1-x^2)^{3/4}]^2\,dx \\
&= 4\pi \int_{-1}^{1} (1-x^2)^{3/2}\,dx \\
&= 4\pi \int_{\arcsin -1 = -\pi/2}^{\arcsin 1 = \pi/2} (1-\sin^2\theta)^{3/2}\cos\theta\,d\theta \qquad [x=\sin\theta, dx = \cos\theta\,d\theta, \theta = \arcsin x] \\
&= 4\pi \int_{-\pi/2}^{\pi/2} \cos^3\theta \cdot \cos\theta\,d\theta \\
&= 8\pi \int_0^{\pi/2} (\cos^2\theta)^2\,d\theta \qquad [\text{Integrand is even}, \cos^4\theta = (\cos^2\theta)^2] \\
&= 8\pi \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\
&= 8\pi \int_0^{\pi/2} \frac{1 + 2\cos 2\theta + \cos^2 2\theta}{4}\,d\theta \\
&= 2\pi \int_0^{\pi/2} 1\,d\theta + 4\pi \int_0^{\pi/2} \cos 2\theta\,d\theta + \pi \int_0^{\pi/2} 1 + \cos 4\theta\,d\theta \\
&= 2\pi\theta\Big|_0^{\pi/2} + 2\pi\sin 2\theta\Big|_0^{\pi/2} + \pi\theta\Big|_0^{\pi/2} + \frac{1}{4}\sin 4\theta\Big|_0^{\pi/2} \\
&= \frac{3\pi^2}{2}.
\end{aligned}
$$

□

5. (10 points) Let R be the region bounded by $y = \dfrac{e^x}{2x}\sin 2x$ and x-axis on $[0, \pi/2]$. Indicate (by shading) the region R in the graph. Determine the volume of the solid of revolution obtained when R is revolved about the y-axis.



Solution. Shell method.

$$
\begin{aligned}
V = \int_a^b 2\pi x f(x)\,dx &= \int_0^{\pi/2} 2\pi x \frac{e^x}{2x}\sin 2x\,dx \\
&= \pi \int_0^{\pi/2} e^x \sin 2x\,dx \quad [u = \sin 2x, dv = e^x dx \implies v = e^x] \\
&= \pi \left(e^x \sin 2x \Big|_0^{\pi/2} - \int_0^{\pi/2} e^x \cdot 2\cos 2x\,dx \right) \quad [\int u\,dv = uv - \int v\,du] \\
&= \pi \left(0 - 2\int_0^{\pi/2} e^x \cos 2x\,dx \right) \quad [u = \cos 2x, dv = e^x dx \implies v = e^x] \\
&= -2\pi \left[e^x \cos 2x \Big|_0^{\pi/2} - \int_0^{\pi/2} e^x \cdot (-2\sin 2x)\,dx \right] \\
&= -2\pi(e^{\pi/2}\cos\pi - e^0 \cos 0) - 4\pi \int_0^{\pi/2} e^x \sin 2x\,dx \\
&= 2\pi(e^{\pi/2} + 1) - 4\pi \int_0^{\pi/2} e^x \sin 2x\,dx.
\end{aligned}
$$

Then we have

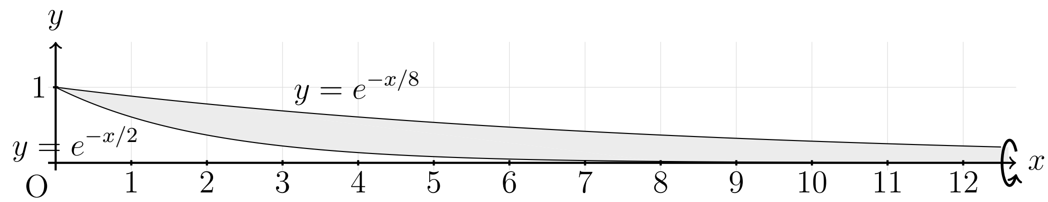
$$5\pi \int_0^{\pi/2} e^x \sin 2x\,dx = 2\pi(e^{\pi/2} + 1).$$

Solve for $\pi \int_0^{\pi/2} e^x \sin 2x\,dx$, we have the volume of the solid as follows

$$V = \pi \int_0^{\pi/2} e^x \sin 2x\,dx = \frac{2\pi}{5}(e^{\pi/2} + 1).$$

□

6. (10 points) Let R be the region bounded by the graph of $y = e^{-x/2}$ and $y = e^{-x/8}$, for $x \geq 0$. Indicate (by shading) the region R in the graph. What is the volume of the solid generated when R is revolved about the x-axis?



SOLUTION. Washer Method. Let $f(x) = e^{-x/8}$, $g(x) = e^{-x/2}$. Then the volume of the solid is

$$
\begin{aligned}
V &= \int_a^b \pi[f(x)^2 - g(x)^2]\,dx \\
&= \int_0^\infty \pi[(e^{-x/8})^2 - (e^{-x/2})^2]\,dx \\
&= \pi \int_0^\infty e^{-x/4} - e^{-x}\,dx \\
&= \pi \lim_{b\to\infty} \int_0^b e^{-x/4} - e^{-x}\,dx \\
&= \pi \lim_{b\to\infty} \int_0^b e^{-x/4}\,dx - \pi \lim_{b\to\infty} \int_0^b e^{-x}\,dx \\
&= \pi \lim_{b\to\infty} \int_{u(0)=0}^{u(b)=-b/4} e^u \cdot (-4)\,du - \pi \lim_{b\to\infty} \int_{v(0)=0}^{v(b)=-b} e^v \cdot (-1)\,dv \quad [u = -\frac{x}{4}, du = -\frac{1}{4}dx, v = -x, dv = \\
&= -4\pi \lim_{b\to\infty} e^u \Big|_0^{-b/4} + \pi \lim_{b\to\infty} e^v \Big|_0^{-b} \\
&= -4\pi \lim_{b\to\infty} (e^{-b/4} - e^0) + \pi \lim_{b\to\infty} (e^{-b} - e^0) \\
&= -4\pi(0-1) + \pi(0-1) \\
&= 3\pi.
\end{aligned}
$$

Therefore, the volume of the solid is 3π. □

7. (10 points) Write $1.\overline{45} = 1.45454545\ldots$ as a geometric series and express its value as fraciton.

Solution.

$$
\begin{aligned}
1.\overline{45} &= 1.45454545\ldots \\
&= 1 + 0.45 + 0.0045 + 0.000045 + \cdots && [a = 0.45, r = \frac{1}{100}, |r|< 1] \\
&= 1 + \sum_{k=0}^{\infty} 0.45 \cdot \left(\frac{1}{100}\right)^k \\
&= 1 + \frac{0.45}{1 - 1/100} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}] \\
&= 1 + \frac{45/100}{99/100} \\
&= 1 + \frac{45}{99} \\
&= 1 + \frac{5}{11} \\
&= \frac{16}{11}.
\end{aligned}
$$

$\square$

8. (15 points) The velocity of an object moving through a fluid can be modeled by the *drag equation*

$$\frac{dv}{dt} = -\frac{1}{16}v^2.$$

(a) Find the general solution to this equation.
(b) An object moving through the water has an initial velocity of 16 m/sec, i.e., $v(0) = 16$. What will the velocity be after 15 seconds?

SOLUTION.

(a) Observe that this differential equation is separable. Let $f(v) = -\frac{1}{16}v^2$, then dividing both sides by $f(v)$ and the integrating with respect to t yields

$$\begin{aligned}
\frac{dv}{dt} &= f(v) \\
\frac{1}{f(v)}\frac{dv}{dt} &= 1 \\
\int \frac{1}{-\frac{1}{16}v^2}\frac{dv}{dt}\,dt &= \int 1\,dt \\
-16\int v^{-2}\,dv &= \int 1\,dt \\
16v^{-1} &= t + C \\
v^{-1} &= \frac{1}{16}(t+C) \\
v &= \frac{16}{t+C}.
\end{aligned}$$

Hence $v = \dfrac{16}{t+C}$ is the general solution to this equation.

(b) Given that $v(0) = 16$, then we have

$$v(0) = \frac{16}{0+C} = 16 \implies C = 1.$$

Then the solution to the initial value problem is

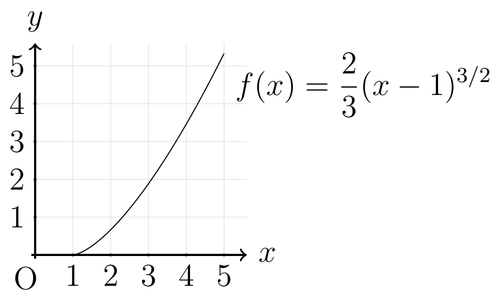
$$v(t) = \frac{16}{t+1}.$$

Hence the velocity after 15 seconds is

$$v(15) = \frac{16}{15+1} = 1 \text{ m/sec}.$$

$\square$

9. (Bonus, 15 points) Approximate the arc length of $f(x) = \frac{2}{3}(x-1)^{3/2}$ on the interval $[1, 5]$ using Trapezoid Rule with $n = 4$ subintervals. What is the absolute error of the approximation?



Solution. Given that $f(x) = \frac{2}{3}(x-1)^{3/2}$, we have $f'(x) = (x-1)^{1/2} \implies f'(x)^2 = (x-1)$. Then the arc length is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_1^5 \sqrt{1 + (x-1)}\, dx \\ &= \int_1^5 \sqrt{x}\, dx \\ &= \frac{2}{3} x^{3/2} \big|_1^5 \\ &= \frac{2}{3} 5^{3/2} - \frac{2}{3}. \end{aligned}$$

Then we approximate the definite integral using Trapezoid Rule as follows. First,

$$\Delta x = \frac{b-a}{n} = \frac{4}{4} = 1, x_k = x_0 + k\Delta x = 1 + k.$$

Let $g(x) = \sqrt{x}$. Then we have

$$\begin{aligned} \int_1^5 \sqrt{x}\, dx = \int_1^5 g(x)\, dx &\approx \left[\frac{1}{2} g(x_0) + \sum_{k=1}^{n-1} g(x_k) + \frac{1}{2} g(x_n)\right] \Delta x \\ &= \left[\frac{1}{2} g(1+0) + \sum_{k=1}^{3} g(1+k) + \frac{1}{2} g(1+4)\right] \cdot 1 \\ &= \left[\frac{1}{2} g(1) + g(2) + g(3) + g(4) + \frac{1}{2} g(5)\right] \\ &= \left[\frac{1}{2}\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \frac{1}{2}\sqrt{5}\right] \\ &\approx 6.7643. \end{aligned}$$

The absolute error is $|c - x| = \left|6.7643 - \left(\frac{2}{3}5^{3/2} - \frac{2}{3}\right)\right| \approx 0.0226.$ $\square$