MATH 2205 - Calculus II Lecture Notes 24

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1 Power Series

1.1 Review of Taylor Polynomials, Taylor/MacLaurin Series

Definition 1.1 (Taylor Polynomials)**.** Let f be a function with $f', f'', \ldots, f^{(n)}$ defined at a. Then nth-order Taylor polynomial for f with its center at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n = f(a), p_n'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} c_k (x-a)^k,$$

where the *coefficients* are

$$c_k = \frac{f^{(k)}(a)}{k!}, \text{for } k = 0, 1, 2, \ldots.$$

Definition 1.2 (Taylor/MacLaurin Series for a Function)**.** Suppose the function f has derivatives of all orders on an interval centered at the point a. The *Taylor series* for f centered at a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a *MacLaurin series*.

Example 1.1 (MacLaurin series for $\sin x$)**.** Find the MacLaurin series for $f(x) = \sin x$, $g(x) = \cos x$, and find their intervals of convergence.

SOLUTION.

(a)

$$\begin{aligned}
f'(x) &= \frac{d}{dx} f(x) = \frac{d}{dx} \sin x = \cos x, \\
f''(x) &= \frac{d}{dx} f'(x) = \frac{d}{dx} \cos x = -\sin x, \\
f^{(3)}(x) &= \frac{d}{dx} f''(x) = \frac{d}{dx}(-\sin x) = -\frac{d}{dx} \sin x = -\cos x, \\
f^{(4)}(x) &= \frac{d}{dx} f^{(3)}(x) = \frac{d}{dx}(-\cos x) = -\frac{d}{dx} \cos x = -(-\sin x) = \sin x.
\end{aligned}$$

Observe that $f^{(4)}(x) = f(x) = \sin(x)$, then $f^{(5)}(x) = f'(x) = \cos x$, $f^{(6)}(x) = f''(x) = -\sin x$, and $f^{(7)}(x) = f^{(3)}(x) = -\cos x$. More generally, $f^{(4n+j)}(x) = f^{(j)}(x), j = 0, 1, 2, 3; n = 0, 1, 2, \ldots$. Then by definition, the MacLaurin series for $f(x) = \sin x$ is

$$\begin{aligned}
\sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
&= \sin 0 + \frac{\cos 0}{1!}x + \frac{-\sin 0}{2!}x^2 + \frac{-\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4 + \frac{\cos 0}{5!}x^5 + \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}
\end{aligned}$$

Then we apply the Ratio Test to $\displaystyle\sum_{k=0}^{\infty} |a_k| = \sum_{k=0}^{\infty} \left| \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right| = \sum_{k=0}^{\infty} \frac{|x|^{2k+1}}{(2k+1)!}$ to test for absolute convergence:

$$\begin{aligned}
r &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\
&= \lim_{k\to\infty} \frac{|x|^{2k+3}/(2k+3)!}{|x|^{2k+1}/(2k+1)!} \\
&= \lim_{k\to\infty} \frac{|x|^2 (2k+1)!}{(2k+1)! \cdot (2k+2)(2k+3)} \\
&= |x|^2 \lim_{k\to\infty} \frac{1}{(2k+2)(2k+3)} \\
&= 0.
\end{aligned}$$

In this case, $r < 1$ for all x, so the MacLaurin series converges absolutely for all x, which implies that the series converges for all x. We conclude that the interval of convergence is $(-\infty, \infty)$.

(b)

$$\begin{aligned}
g'(x) &= \frac{d}{dx} g(x) = \frac{d}{dx} \cos x = -\sin x, \\
g''(x) &= \frac{d}{dx} g'(x) = \frac{d}{dx}(-\sin x) = -\frac{d}{dx}\sin x = -\cos x, \\
g^{(3)}(x) &= \frac{d}{dx} g''(x) = \frac{d}{dx}(-\cos x) = -\frac{d}{dx}\cos x = -(-\sin x) = \sin x, \\
g^{(4)}(x) &= \frac{d}{dx} g^{(3)}(x) = \frac{d}{dx}\sin x = \cos x,
\end{aligned}$$

Observe that $g^{(4)}(x) = g(x) = \cos(x)$, then $g^{(5)}(x) = g'(x) = -\sin x$, $g^{(6)}(x) = g''(x) = -\cos x$, and $g^{(7)}(x) = g^{(3)}(x) = \sin x$. More generally, $g^{(4n+j)}(x) = g^{(j)}(x), j = 0, 1, 2, 3; n =$

$0, 1, 2, \ldots$. Then by definition, the MacLaurin series for $g(x) = \cos x$ is

$$\begin{aligned}
\cos x = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}(x-0)^k &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}x^k \\
&= \cos 0 + \frac{-\sin 0}{1!}x + \frac{-\cos 0}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \frac{\cos 0}{4!}x^4 + \frac{-\sin 0}{5!}x^5 + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}
\end{aligned}$$

Or we can observe that $\cos x = \frac{d}{dx}\sin x$, then

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{d}{dx}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)x^{2k}}{(2k)!\cdot(2k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}.$$

Then we apply the Ratio Test to $\sum_{k=0}^{\infty}|a_k| = \sum_{k=0}^{\infty}\left|\frac{(-1)^k x^{2k}}{(2k)!}\right| = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$ to test for absolute convergence:

$$\begin{aligned}
r &= \lim_{k\to\infty}\frac{a_{k+1}}{a_k} \\
&= \lim_{k\to\infty}\frac{x^{2(k+1)}/[2(k+1)]!}{x^{2k}/(2k)!} \\
&= \lim_{k\to\infty}\frac{x^{2k+2}}{x^{2k}}\frac{(2k)!}{(2k+2)!} \\
&= \lim_{k\to\infty}\frac{x^{2k}\cdot x^2}{x^{2k}}\frac{(2k)!}{(2k)!\,(2k+1)(2k+2)} \\
&= x^2\lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)} \\
&= 0.
\end{aligned}$$

In this case, $r < 1$ for all x, so the MacLaurin series converges absolutely for all x, which implies that the series converges for all x. We conclude that the interval of convergence is $(-\infty, \infty)$.

□

Example 1.2 (Manipulating MacLaurin series)**.** Let $f(x) = e^x$.

(a) Find the MacLaurin series for f.
(b) Find its interval of convergence.
(c) Use the MacLaurin series for e^x to find the MacLaurin series for the functions x^4e^x, e^{-2x}, and e^{-x^2}.

Solution.

(a) If $f(x) = e^x$, then $f^{(k)}(x) = f^{(k-1)}(x) = \cdots = f''(x) = f'(x) = f(x) = e^x$. Hence $f^{(k)}(0) = f^{(k-1)}(0) = \cdots = f''(0) = f'(0) = f(0) = e^0 = 1$. Given center $a = 0$, we have

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{e^0}{k!}x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

(b) Then we apply the Ratio Test to $\sum_{k=0}^{\infty}|a_k| = \sum_{k=0}^{\infty}\left|\frac{x^k}{k!}\right| = \sum_{k=0}^{\infty}\frac{|x|^k}{k!}$ to test for absolute convergence:

$$\begin{aligned} r &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\ &= \lim_{k\to\infty} \frac{|x|^{k+1}/(k+1)!}{|x|^k/k!} \\ &= \lim_{k\to\infty} |x| \frac{k!}{k! \cdot (k+1)} \\ &= |x| \lim_{k\to\infty} \frac{1}{k+1} \\ &= 0. \end{aligned}$$

In this case, $r < 1$ for all x, so the MacLaurin series converges absolutely for all x, which implies that the series converges for all x. We conclude that the interval of convergence is $(-\infty, \infty)$.

(c) Apply the Theorem of Combining Power Series, we have the following

$$\begin{aligned} x^4 e^x &= x^4 \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k x^4}{k!} = \sum_{k=0}^{\infty} \frac{x^{k+4}}{k!}, \\ e^{-2x} &= \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^k}{k!}, \\ e^{-x^2} &= \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}. \end{aligned}$$

□

Proposition 1.1 (MacLaurin series)**.**

(a) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, for $|x| < \infty$.

(b) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, for $|x| < \infty$.

(c) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, for $|x| < \infty$.

(d) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, for $|x| < 1$.

(e) $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$, for $|x| < 1$.

(f) $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$, for $-1 < x \leq 1$.

(g) $-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$, for $-1 \leq x < 1$.

1.2 Remainder and Approximation of Series

Definition 1.3 (Remainder)**.** The *remainder* is the error in approximating a convergent series by the sum of its first n terms, that is,

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \cdots.$$

Theorem 1.1 (Estimating Series with Positive Terms)**.** Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n < \int_n^{\infty} f(x)\,dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\,dx.$$

Example 1.3 (Approximating a p-series)**.**

(a) How many terms of the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

(b) Find an approximation to the series using 50 terms of the series.

Solution. The function associated with this series is $f(x) = \frac{1}{x^2}$.

(a) Using the bound on the remainder, we have

$$\begin{aligned} R_n < \int_n^{\infty} f(x)\,dx &= \int_n^{\infty} \frac{dx}{x^2} \\ &= \lim_{b\to\infty} \int_n^b \frac{dx}{x^2} \\ &= \lim_{b\to\infty} -\frac{1}{x}\bigg|_n^b \\ &= \lim_{b\to\infty} -\left(\frac{1}{b} - \frac{1}{n}\right) \\ &= \frac{1}{n}. \end{aligned}$$

(b) To ensure that $R_n < 10^{-3}$, we must choose n so that $\frac{1}{n} < 10^{-3}$, which implies $n > 1000$. In otherwords, we must sum at least 1001 terms of the series to be sure that the remainder is less than 10^{-3}.

(c) Using the bounds on the series, we have $L_m < S < U_m$, where S is the exact value of the series, and

$$L_n = S_n + \int_{n+1}^{\infty} \frac{dx}{x^2} = S_n + \frac{1}{n+1} \text{ and } U_n = S_n + \int_{n}^{\infty} \frac{dx}{x^2} = S_n + \frac{1}{n}.$$

Therefore, the series is bounded as follows,

$$S_n + \frac{1}{n+1} < S < S_n + \frac{1}{n},$$

where S_n is the sum of the first n terms. Usinga a calculator to sum the first 50 terms of the series, we find $S_{50} \approx 1.625133$. The exact value of the series is in the interval

$$S_{50} + \frac{1}{51} < S < S_{50} + \frac{1}{50}.$$

or $1.644741 < S < 1.645133$. Taking the average of these two bounds as out approximation of S, we find that $S \approx 1.644937$.

$\square$

Theorem 1.2 (Remainder in Alternating Series)**.** Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \leq a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Example 1.4 (Remainder in an alternating series)**.**

(a) In turns out that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(1+x)\Big|_{x=1} = \sum_{k=0}^{\infty} \frac{(-1)^k 1^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$. How many terms of the series are required to approximate $\ln 2$ with an error less than 10^{-6}? The exact value of the series is given but is not needed to answer the question.

(b) Consider the series $-1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$. Find an upper bound for the magnitude of the error in approximating the value of the series (which is $e^{-1} - 1$) with $n = 9$ terms.

Solution.

(a) The series is expressed as the sum of the first n terms plus the remainder:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \frac{(-1)^{n+1}}{n+2} + \cdots.$$

In magnitude, the remainder is less than or equal to the magnitude of the $(n+1)$st term:

$$|R_n| = |S - S_n| \leq a_{n+1} = \frac{1}{n+2}.$$

To ensure that the error is less than 10^{-6}, we require that

$$a_{n+1} = \frac{1}{n+2} < 10^{-6} \implies n + 2 > 10^6 \implies n > 10^6 - 2.$$

Therefore, it takes 1 million terms of the series to approximate $\ln 2$ with an error less than 10^{-6}.

(b) The series may be expressed as the sum of the first nine terms plus the remainder:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} = -1 + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{9!} + \frac{1}{10!} - \cdots.$$

The error committed when using the first nine terms to approximate the value of the series satisfies

$$|R_9| = |S - S_9| \leq a_{10} = \frac{1}{10!} \approx 2.8 \times 10^{-7}.$$

Therefore, the error is no greater than 2.8×10^{-7}. As a check, the difference between the sum of the first nine terms, $\sum_{k=1}^{9} \frac{(-1)^k}{k!} \approx -0.632120811$, and the exact value, $S = e^{-1} - 1 \approx -0.632120559$, is approximately 2.5×10^{-7}.

$\square$