MATH 2205 - Calculus II Lecture Notes 23

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1 Power Series

1.1 Review of Combining, Differentiating and Integrating Power Series

Theorem 1.1 (Convergence of Power Series)**.** A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

(a) The series converges for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
(b) There is a real number $R > 0$ such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$, in which case the radius of convergence is R.
(c) The series converges only at a, in which case the radius of convergence is $R = 0$.

Theorem 1.2 (Combining Power Series)**.** Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

(a) *Sum and difference*: The power series $\sum(c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I.
(b) *Multiplication by a power*: Suppose m is an integer such that $k + m \geq 0$ for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x\to 0} x^m f(x)$.
(c) *Composition*: If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k(h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Theorem 1.3 (Differentiating and Integrating Power Series)**.** Suppose the power series $\sum c_k(x-a)^k$ converges for $|x - a| < R$ and defines a function f on that interval.

(a) Then f is differentiable (which implies continuous) for $|x - a| < R$, and f' is found by differentiating the power series for f term by term: that is,

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\sum c_k(x-a)^k = \sum \frac{d}{dx}c_k(x-a)^k = \sum kc_k(x-a)^{k-1},$$

for $|x - a| < R$.
(b) The indefinite integral of f is found by integrating the power series for f term by term: that is,

$$\int f(x)\,dx = \int \sum c_k(x-a)^k\,dx = \sum \int c_k(x-a)^k\,dx = \sum c_k\frac{(x-a)^{k+1}}{k+1} + C,$$

for $|x - a| < R$, where C is an arbitrary constant.

Example 1.1 (Differentiating and integrating power series)**.** Consider the geometric series

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \text{for } |x| < 1.$$

(a) Differentiate this series term by term to find the power series for f' and identify the function it represents.
(b) Integrate this series term by term and identify the function it represents.

Solution. (a)

$$\begin{aligned} f'(x) &= \frac{d}{dx} f(x) \\ &= \frac{d}{dx}\frac{1}{1-x} \\ &= \frac{d}{dx}\sum_{k=0}^{\infty} x^k \\ &= \sum_{k=0}^{\infty} \frac{d}{dx} x^k \\ &= \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{x}x^2 + \frac{d}{dx}x^3 + \cdots \\ &= 0 + 1 + 2x + 3x^2 + \cdots \\ &= \sum_{k=0}^{\infty} (k+1)x^k. \end{aligned}$$

It is known that

$$\begin{aligned} f'(x) &= \frac{d}{dx}\frac{1}{1-x} \\ &= \frac{d}{dx}(1-x)^{-1} \\ &= (-1)(1-x)^{-2} \cdot (-1) \\ &= (1-x)^{-2} \\ &= \frac{1}{(1-x)^2}. \end{aligned}$$

Then we can conclude that

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k.$$

(b)

$$\begin{aligned} \int f(x)\,dx &= \int \frac{1}{1-x}\,dx \\ &= \int \sum_{k=0}^{\infty} x^k\,dx \\ &= \sum_{k=0}^{\infty} \int x^k\,dx \\ &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C. \end{aligned}$$

And we can use change of variables to find the integral of $f(x) = \frac{1}{1-x}$ as follows,

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{1}{1-x}\,dx \\
&= \int (1-x)^{-1}\,dx \\
&= -\int u^{-1}\,du \qquad\qquad [u = 1-x, du = -dx \implies dx = -du] \\
&= -\ln u + C \\
&= -\ln(1-x) + C.
\end{aligned}$$

When $x = 1$, $C = 0$. Therefore,

$$-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

$\square$

Example 1.2 (Functions to power series)**.** Find power series representations centered at 0 for the following functions and give their intervals of convergence.

(a) $\arctan x$.

(b) $\ln\left(\dfrac{1+x}{1-x}\right)$.

Solution.

(a) Recall that

$$\begin{aligned}
\arctan x = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int \sum_{k=0}^{\infty}(-x^2)^k\,dx &= \int \sum_{k=0}^{\infty}(-1)^k x^{2k}\,dx \\
&= \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\,dx \\
&= \sum_{k=0}^{\infty}(-1)^k \int x^{2k}\,dx \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} + C.
\end{aligned}$$

Substituting $x = 0$ and noting that $\arctan 0 = 0$, the two sides of this equation agree provided we choose $C = 0$. Therefore, $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$. And the interval of convergence is same as $\frac{1}{1+x^2}$, therefore, $|x^2| < 1 \implies -1 < x < 1$ (or we can use Ratio Test, which gives the same result).

(b) Observe that $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_{k=0}^{\infty}(-x)^k$ and $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x} = -\sum_{k=0}^{\infty} x^k$, then

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) &= \int \frac{1}{1+x}\,dx - \left(-\int \frac{1}{1-x}\,dx\right) \\
&= \int \frac{1}{1-(-x)}\,dx + \int \frac{1}{1-x}\,dx \\
&= \int \sum_{k=0}^{\infty}(-x)^k\,dx + \int \sum_{k=0}^{\infty} x^k\,dx \\
&= \sum_{k=0}^{\infty}\int (-1)^k x^k\,dx + \sum_{k=0}^{\infty}\int x^k\,dx \\
&= \sum_{k=0}^{\infty}(-1)^k \int x^k\,dx + \sum_{k=0}^{\infty}\int x^k\,dx \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} + \sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1} + C \\
&= \sum_{k=0}^{\infty}\frac{2x^{2k+1}}{k+1} + C \\
&= 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{k+1} + C.
\end{aligned}$$

The power series is the difference of two power series, both of which converge on the interval $|x| < 1$. Therefore, the new series also converge on $|x| < 1$.

□

Definition 1.1 (Taylor Polynomials)**.** Let f be a function with $f', f'', \ldots, f^{(n)}$ defined at a. Then nth-order Taylor polynomial for f with its center at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n = f(a), p_n'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} c_k(x-a)^k,$$

where the *coefficients* are

$$c_k = \frac{f^{(k)}(a)}{k!}, \text{for } k = 0, 1, 2, \ldots.$$

Definition 1.2 (Taylor/MacLaurin Series for a Function)**.** Suppose the function f has derivatives of all orders on an interval centered at the point a. The *Taylor series* for f centered at a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a *MacLaurin series*.

Example 1.3 (MacLaurin series and convergence)**.** Find the MacLaurin series (which is the Taylor series centered at 0) for $f(x) = \dfrac{1}{1-x}$. Find the interval of convergence.

Solution. We can find the derivatives of $f(x) = 1/(1-x) = (1-x)^{-1}$ as follows:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}\cdot(-1) = (1-x)^{-2},\\
f''(x) &= \frac{d}{dx}(1-x)^{-2} = (-2)(1-x)^{-3}\cdot(-1) = 2(1-x)^{-3},\\
f'''(x) &= \frac{d}{dx}2(1-x)^{-3} = 2\cdot(-3)(1-x)^{-4}\cdot(-1) = 2\cdot 3(1-x)^{-4} = 3!\,(1-x)^{-4},\\
f^{(4)}(x) &= \frac{d}{dx}3!\,(1-x)^{-4} = 3!\cdot(-4)(1-x)^{-5}\cdot(-1) = 4!\,(1-x)^{-5},\\
f^{(5)}(x) &= \frac{d}{dx}4!\,(1-x)^{-5} = 4!\cdot(-5)(1-x)^{-6}\cdot(-1) = 5!\,(1-x)^{-6},\\
&\ \ \vdots\\
f^{(k)}(x) &= k!\,(1-x)^{-k-1},\\
&\ \ \vdots
\end{aligned}$$

Therefore, the MacLaurin series for f is

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{k!\,(1-0)^{-k-1}}{k!}x^k = \sum_{k=0}^{\infty} x^k.$$

The MacLaurin series for $f(x) = 1/(1-x)$ is a geometric series. We could apply the Ratio Test, but we have already demonstrated that this series converges for $|x| < 1$. $\square$