MATH 2205 - Calculus II Lecture Notes 22

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1 Sequences and Series

1.1 Review of Convergent Series and Power Series

Theorem 1.1 (Properties of Convergent Series)**.**

(a) Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.
(b) Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
(c) If M is a positive integer, then $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Definition 1.1 (Absolute and Conditional Convergence)**.** If $\sum|a_k|$ converges, then $\sum a_k$ *converges absolutely.* If $\sum|a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ *converges conditionally.*

Theorem 1.2 (Absolute Convergence Implies Convergence)**.** If $\sum|a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum|a_k|$ diverges.

Definition 1.2 (Power Series)**.** A *power series* has the general form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the *coefficients* of the power series and a is the *center* of the power series. The set of values of x for which the series converges is its *interval of convergence.* The *radius of convergence* of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Theorem 1.3 (Convergence of Power Series)**.** A power series $\sum\limits_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

(a) The series converges for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
(b) There is a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is R.
(c) The series converges only at a, in which case the radius of convergence is $R = 0$.

Example 1.1 (Interval and radius of convergence)**.** Use the Ratio Test to find the radius and interval of convergence of $\sum\limits_{k=1}^{\infty} \dfrac{(x-2)^k}{\sqrt{k}}$.

Solution.

$$\begin{aligned} r &= \lim_{k\to\infty} \frac{|(x-2)^{k+1}/\sqrt{k+1}|}{|(x-2)^k/\sqrt{k}|} \\ &= \lim_{k\to\infty} |x-2| \frac{\sqrt{k}}{\sqrt{k+1}} \\ &= |x-2| \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^{1/2} \\ &= |x-2| \left(\lim_{k\to\infty} \frac{1}{1+1/k}\right)^{1/2} \\ &= |x-2|. \end{aligned}$$

The series converges absolutely (and therefore converges) for all x such that $r < 1$, which implies $|x-2| < 1$, or $1 < x < 3$. On the interval $-\infty < x < 1$ and $3 < x < \infty$, we have $r > 1$ and the series diverges. We now test the endpoints. Substitute $x = 1$ gives the series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}},$$

which converges by the Alternating Series Test. Substituting $x = 3$ gives the series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}},$$

which is a p-series with $p = 1/2$, therefore, it diverges. We conclude that the interval of convergence is $1 \le x < 3$ and the radius of convergence is $R = 1$. □

Theorem 1.4 (Combining Power Series)**.** Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

(a) *Sum and difference*: The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I.
(b) *Multiplication by a power*: Suppose m is an integer such that $k + m \ge 0$ for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x\to 0} x^m f(x)$.
(c) *Composition*: If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Example 1.2 (Combining power series)**.** Given the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \text{for } |x| < 1,$$

find the power series and interval of convergence for the following functions.

(a) $\dfrac{x^5}{1-x}$.
(b) $\dfrac{1}{1-2x}$.

(c) $\dfrac{1}{1+x^2}$.

Solution.

(a) By Theorem 1.4, we have

$$\frac{x^5}{1-x} = x^5 \frac{1}{1-x} = x^5 \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^5 x^k = \sum_{k=0}^{\infty} x^{k+5}.$$

This geometric series has a ratio $r = x$ and converges when $|r| = |x| < 1$. The interval of convergence is $|x| < 1$.

(b) By Theorem 1.4, we substitute $2x$ for x in the power series for $\dfrac{1}{1-x}$:

$$\frac{1}{1-2x} = \sum_{k=0}^{\infty} (2x)^k.$$

This geometric series has a ratio $r = 2x$ and converges provided $|r| = |2x| < 1$ or $|x| < \dfrac{1}{2}$. The interval of convergence is $|x| < \dfrac{1}{2}$.

(c) We substitute $-x^2$ for x in the power series for $\frac{1}{1-x}$:

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1 \cdot x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

This geometric series has a ratio of $r = -x^2$ and converges provided $|r| = |-x^2| = |x^2| < 1$ or $|x| < 1$.

$\square$

Theorem 1.5 (Differentiating and Integrating Power Series)**.** Suppose the power series $\sum c_k (x-a)^k$ converges for $|x-a| < R$ and defines a function f on that interval.

(a) Then f is differentiable (which implies continuous) for $|x-a| < R$, and f' is found by differentiating the power series for f term by term: that is,

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum c_k (x-a)^k = \sum \frac{d}{dx} c_k (x-a)^k = \sum k c_k (x-a)^{k-1},$$

for $|x-a| < R$.

(b) The indefinite integral of f is found by integrating the power series for f term by term: that is,

$$\int f(x)\,dx = \int \sum c_k (x-a)^k\,dx = \sum \int c_k (x-a)^k\,dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$

for $|x-a| < R$, where C is an arbitrary constant.

Example 1.3 (Differentiating and integrating power series)**.** Consider the geometric series

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \text{for } |x| < 1.$$

(a) Differentiate this series term by term to find the power series for f' and identify the function it represents.
(b) Integrate this series term by term and identify the function it represents.

Solution. (a)

$$\begin{aligned} f'(x) &= \frac{d}{dx} f(x) \\ &= \frac{d}{dx}\frac{1}{1-x} \\ &= \frac{d}{dx}\sum_{k=0}^{\infty} x^k \\ &= \sum_{k=0}^{\infty} \frac{d}{dx} x^k \\ &= \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{x}x^2 + \frac{d}{dx}x^3 + \cdots \\ &= 0 + 1 + 2x + 3x^2 + \cdots \\ &= \sum_{k=0}^{\infty}(k+1)x^k. \end{aligned}$$

It is known that

$$\begin{aligned} f'(x) &= \frac{d}{dx}\frac{1}{1-x} \\ &= \frac{d}{dx}(1-x)^{-1} \\ &= (-1)(1-x)^{-2}\cdot(-1) \\ &= (1-x)^{-2} \\ &= \frac{1}{(1-x)^2}. \end{aligned}$$

Then we can conclude that

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k.$$

(b)

$$\begin{aligned} \int f(x)\,dx &= \int \frac{1}{1-x}\,dx \\ &= \int \sum_{k=0}^{\infty} x^k\,dx \\ &= \sum_{k=0}^{\infty}\int x^k\,dx \\ &= \sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1} + C. \end{aligned}$$

And we can use change of variables to find the integral of $f(x) = \frac{1}{1-x}$ as follows,

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{1}{1-x}\,dx \\
&= \int (1-x)^{-1}\,dx \\
&= -\int u^{-1}\,du && [u = 1-x, du = -dx \implies dx = -du] \\
&= -\ln u + C \\
&= -\ln(1-x) + C.
\end{aligned}$$

When $x = 1$, $C = 0$. Therefore,

$$-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

$\square$