MATH 2205 - Calculus II Lecture Notes 20

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1 Sequences and Series

1.1 Review of p-Series, Ratio, and Root Tests

Theorem 1.1 (Convergence of the p-Series)**.** The *p-series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 1.2 (Ratio Test)**.** Let $\sum a_k$ be an infinite series with positive terms and let $r = \displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

(a) If $0 \leq r < 1$, the series converges.
(b) If $r > 1$ (including $r = \infty$), the series diverges.
(c) If $r = 1$, the test is inconclusive.

Theorem 1.3 (Root Test)**.** Let $\sum a_k$ be an infinite series with nonnegative terms and let $\rho = \displaystyle\lim_{k\to\infty} \sqrt[k]{a_k}$.

(a) If $0 \leq \rho < 1$, the series converges.
(b) If $\rho > 1$ (including $\rho = \infty$), the series diverges.
(c) If $\rho = 1$, the test is inconclusive.

1.2 Comparison, Limit Comparison, Alternating Series Tests

Theorem 1.4 (Comparison Test)**.** Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

(a) If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(b) If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 1.1 (Using the Comparison Test)**.** Determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^4 - 1}$.
(b) $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{k^3}$.

Solution.

(a) Observe that $\dfrac{k^3}{2k^4 - 1} > \dfrac{k^3}{2k^4} = \dfrac{1}{2k}$. Because

$$\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$$

is half of the Harmonic series which diverges. Then by Comparison Test, the given series also diverges.

(b) Note that $\ln k < k$, for $k \geq 2$, and then divide by k^3, we have

$$\frac{\ln k}{k^3} < \frac{k}{k^3} = \frac{1}{k^2}.$$

Therefore, an appropriate comparison series is the convergent p-series $\sum_{k=2}^{\infty} \frac{1}{k^2}$. Because $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, the given series converges.

□

Theorem 1.5 (Limit Comparison Test)**.** Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

(a) If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
(b) If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(c) If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 1.2 (Using the Limit Comparison Test)**.** Determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$.
(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$.

Solution.

(a) As $k \to \infty$, the rational function behaves like the ratio of the leading (highest-power) terms. In this case, as $k \to \infty$,

$$\frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5} \approx \frac{5k^4}{2k^6} = \frac{5}{2k^2}.$$

Therefore, a reasonable comparison series is the convergenct p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Having chosen a comparison series, we compute the limit L:

$$\begin{aligned}
L &= \lim_{k\to\infty} \frac{a_k}{b_k} \\
&= \lim_{k\to\infty} \frac{(5k^4 - 2k^2 + 3)/(2k^6 - k + 5)}{1/k^2} \\
&= \lim_{k\to\infty} \frac{k^2(5k^4 - 2k^2 + 3)}{2k^6 - k + 5} \\
&= \lim_{k\to\infty} \frac{5k^6 - 2k^4 + 3k^2}{2k^6 - k + 5} \\
&= \lim_{k\to\infty} \frac{5 - 2k^{-2} + 3k^{-4}}{2 - k^{-5} + 5k^{-6}} \\
&= \frac{5}{2}.
\end{aligned}$$

We see that $0 < L = \frac{5}{2} < \infty$, therefore, the given series converges.

(b) Note that $1 < \ln k < k$ as $k \to \infty$, that implies

$$\frac{1}{k^2} < \frac{\ln k}{k^2} < \frac{k}{k^2}.$$

Let $a_k = \frac{\ln k}{k^2}$ and $b_k = \frac{1}{k^2}$, then

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \ln k = \infty.$$

The Limit Comparison Test does not apply because the comparison series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we can reach the conclusion only when the comparison series diverges. If, instead, we use the comparison series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$, then

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k^2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\ln k}{k} = \lim_{k\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Again, the Limit Comparison Test does not apply because the comparison series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we can reach the conclusion only when the comparison series converges. Hence we need a series that lies "between" $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ is the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$, we try it as a comparison series. Let $a_k = \frac{\ln k}{k^2}$ and $b_k = \frac{1}{k^{3/2}}$, we find that

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k^2}}{\frac{1}{k^{3/2}}} = \lim_{k\to\infty} \frac{\ln k}{k^{1/2}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{1/x}{1/2x^{-1/2}} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0.$$

The Limit Comparison Test applies, the comparison series $\sum_{k=1}^{\infty} \frac{1}{x^{3/2}}$ converges, so the given series converges.

□

Theorem 1.6 (Alternating Series Test)**.** The alternating series $\sum(-1)^{k+1}a_k$ converges provided

(a) the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \le a_k$), for k greater than some index N) and

(b) $\lim_{k\to\infty} a_k = 0$.

Theorem 1.7 (Alternating Harmonic Series)**.** The alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges).

Example 1.3 (Alternating Series Test)**.** Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

(b) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$.

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^{k} \ln k}{k}$.

SOLUTION.

(a) We can identify that $a_k = \frac{1}{k^2}$, which is nonincreasing and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^2} = 0$. Therefore, by Alternating Series Test, the given series converges.

(b)

$$\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k}.$$

Therefore, $a_k = \frac{k+1}{k} = 1 + \frac{1}{k}$. And

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} 1 + \frac{1}{k} = 1.$$

Hence the Alternating Series Test is inconlusive. However, by Divergence Test, $\lim_{k\to\infty} (-1)^{k+1} a_k \neq 0$, then the series diverges.

(c) In this series, $a_k = \frac{\ln k}{k}$, whose magnitude is decreasing, and

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln k}{k} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, by Alternating Series Test, the series converges.

□

Theorem 1.8 (Growth Rates of Sequences)**.** The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s and $b > 1$.

Procedure 1.1 (Guidelines for Choosing a Test)**.**

(a) Begin with Divergence Test. If you show that $\lim_{k\to\infty} a_k \neq 0$, then the series diverges and your work is finished. The order of growth rates of sequences is useful for evaluating $\lim_{k\to\infty} a_k$.

(b)
- Geometric series: $\sum ar^k$ converges for $|r| < 1$ and diverges for $|r| \geq 1 (a \neq 0)$.
- p-series: $\sum \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
- Check also for a telescoping series.

(c) If the genearl kth term of the series looks like a function you can integrate, then try the Integral Test.

(d) If the general kth term of the series involves $k!$, k^k, a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.

(e) If the general kth term of the series is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test.

(f) If the sign of the terms is alternating, use the Alternating Series Test.