# MATH 2205 - Calculus II Lecture Notes 19

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## 1 Sequences and Series

### 1.1 Review of The Divergence and Integral Tests

**Theorem 1.1** (Contrapositive and Converse)**.** If the statement "if $p$, then $q$" (i.e., $p \implies q$) is true, then its *contrapositive*, "if (not $q$), then (not $p$)" (i.e., $\neg q \implies \neg p$), is also true. However its *converse*, "if $q$, then $p$" (i.e., $q \implies p$), is not necessary true. In short,

$$p \implies q \equiv \neg q \implies \neg p,$$
$$p \implies q \not\equiv q \implies p,$$

where $A \equiv B$ means $A$ and $B$ are equivalent.

**Theorem 1.2** (Divergence Test)**.** If $\sum a_k$ converges, then $\lim\limits_{k\to\infty} a_k = 0$. Equivalently, if $\lim\limits_{k\to\infty} a_k \neq 0$, then the series diverges.

Note: The converse of the above statement, if $\lim\limits_{k\to\infty} a_k = 0$, then the series converges, might not be true.

**Theorem 1.3** (Harmonic Series)**.** The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges – even though the terms of the series approach zero.

**Proposition 1.1.**

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \approx \lim_{n\to\infty} \ln n + \gamma,$$

where $\gamma \approx 0.57721\ldots$.

**Theorem 1.4** (Integral Test)**.** Suppose $f$ is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

**Example 1.1** (Applying the Integral Test)**.** Determine whether the following series converge.

(a) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\sqrt{2k-5}}$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2+4}$.

SOLUTION.

(a) In this case, the relevant integral is

$$\begin{aligned}\int_3^\infty \frac{dx}{\sqrt{2x-5}} &= \lim_{b\to\infty}\int_3^b \frac{dx}{\sqrt{2x-5}} \\ &= \lim_{b\to\infty}\frac{1}{2}\int_{u(3)}^{u(b)} u^{-1/2}\,du \qquad [u = 2x-5, du = 2\,dx \implies dx = \frac{1}{2}\,du] \\ &= \frac{1}{2}\lim_{b\to\infty} 2u^{1/2}\Big|_{u(3)=1}^{u(b)=2b-5} \\ &= \lim_{b\to\infty}(2b-5)^{1/2}-1 \\ &= \infty.\end{aligned}$$

Therefore, the given series diverges.

(b) The function associated with this series is $f(x) = \dfrac{1}{x^2+4}$, which is positive, decreasing for $x \geq 0$. The integral that determines convergence is

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+4}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{x^2+4}\,dx \\ &= \lim_{b\to\infty}\int_0^b \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx \\ &= \frac{1}{4}\lim_{b\to\infty}\int_{u(0)=0}^{u(b)=b/2}\frac{2}{1+u^2}\,du \qquad [u = \frac{x}{2}, du = \frac{1}{2}\,dx \implies dx = 2\,du] \\ &= \frac{1}{2}\lim_{b\to\infty}\int_0^{b/2}\frac{1}{1+u^2}\,du \\ &= \frac{1}{2}\lim_{b\to\infty}\arctan u\Big|_0^{b/2} \\ &= \frac{1}{2}\lim_{b\to\infty}(\arctan b/2 - \arctan 0) \\ &= \frac{1}{2}(\frac{\pi}{2}-0) \\ &= \frac{\pi}{4}.\end{aligned}$$

Because the integral is finite (equivalently, it converges), the infinite series also converges (but not to $\frac{\pi}{4}$).

□

**Theorem 1.5** (Convergence of the $p$-Series)**.** The *$p$-series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

*Proof.* The function associated with this series is $f(x) = \frac{1}{x^p}$. The integral determines convergence is

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx \\
&= \lim_{b\to\infty} \int_1^b x^{-p}\,dx \\
&= \begin{cases} \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b & \text{if } p \neq 1 \\ \lim_{b\to\infty} \ln|x|\Big|_1^b & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} & \text{if } p \neq 1 \\ \lim_{b\to\infty} \ln p - 0 & \text{if } p = 1 \end{cases} \\
&= \begin{cases} 0 - \frac{1}{-p+1} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \\ \infty & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}
\end{aligned}$$

□

**Example 1.2** (Using the $p$-series test)**.** Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} k^{-3}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}}$.

(c) $\displaystyle\sum_{k=4}^{\infty} \frac{1}{(k-1)^2}$.

SOLUTION.

(a) Because $\displaystyle\sum_{k=1}^{\infty} k^{-3} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a $p$-series with $p = 3$, it converges by Theorem 1.5.

(b) Because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/4}}$ is a $p$-series with $p = \frac{3}{4}$, it diverges 1.5.

(c) The series

$$\sum_{k=4}^{\infty} \frac{1}{(k-1)^2} = \sum_{k=3}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{2} \frac{1}{k^2}.$$

The first sum is $p$-series with $p = 2$, which converges by Theorem 1.5, while the second sum is just a finite number. Therefore, the series $\displaystyle\sum_{k=4}^{\infty} \frac{1}{(k-1)^2}$ converges.

□

## 1.2 The Ratio, Root, and Comparison Tests

**Theorem 1.6** (Ratio Test)**.** Let $\sum a_k$ be an infinite series with positive terms and let $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

(a) If $0 \leq r < 1$, the series converges.
(b) If $r > 1$ (including $r = \infty$), the series diverges.
(c) If $r = 1$, the test is inconclusive.

**Example 1.3** (Using the Ratio Test)**.** Use the Ratio Test to determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$.
(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$.
(c) $\displaystyle\sum_{k=1}^{\infty} e^{-k}(k^2+4)$.

Solution.

(a)

$$\begin{aligned} r &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\ &= \lim_{k\to\infty} \frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}} \\ &= \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \frac{k!}{10^k} \\ &= \lim_{k\to\infty} \frac{10}{k+1} \\ &= 0. \end{aligned}$$

By the Ratio Test, the series converges because $r = 0 < 1$.

(b)

$$
\begin{aligned}
r &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\
&= \lim_{k\to\infty} \frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}} \\
&= \lim_{k\to\infty} \frac{(k+1)^k}{k!}\frac{k!}{k^k} \\
&= \lim_{k\to\infty} \frac{(k+1)^k}{k^k} \\
&= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k \\
&= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k \\
&= \lim_{k\to\infty} e^{k\ln\left(1+\frac{1}{k}\right)} && [a^b = e^{b\ln a}] \\
&= e^{\lim_{k\to\infty} k\ln\left(1+\frac{1}{k}\right)} && [\lim_{k\to\infty} k\ln\left(1+\frac{1}{k}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}] \\
&= e^1 && [\text{L'Hôpital's Rule}: \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\frac{-1}{x^2}}{-\frac{1}{x^2}} = 1] \\
&= e.
\end{aligned}
$$

By the Ratio Test, the series diverges because $r = e > 1$.

(c)

$$
\begin{aligned}
r &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\
&= \lim_{k\to\infty} \frac{e^{-(k+1)}[(k+1)^2+4]}{e^{-k}(k^2+4)} \\
&= \lim_{k\to\infty} \frac{e^{-1}[(k+1)^2+4]}{k^2+4} \\
&= e^{-1}\lim_{k\to\infty} \frac{1+\frac{4}{(k+1)^2}}{\frac{k^2}{(k+1)^2}+\frac{4}{(k+1)^2}} \\
&= e^{-1}\lim_{k\to\infty} \frac{1+\frac{4}{(k+1)^2}}{\frac{1}{(1+1/k)^2}+\frac{4}{(k+1)^2}} \\
&= e^{-1}\frac{1+0}{1+0} \\
&= e^{-1}.
\end{aligned}
$$

By the Ratio Test, the series converges because $r = e^{-1} < 1$.

□

**Theorem 1.7** (Root Test)**.** Let $\sum a_k$ be an infinite series with nonnegative terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.

(a) If $0 \le \rho < 1$, the series converges.
(b) If $\rho > 1$ (including $\rho = \infty$), the series diverges.
(c) If $\rho = 1$, the test is inconclusive.

**Example 1.4** (Using the Root Test)**.** Use the Root Test to determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{4k^2 - 3}{7k^2 + 6}\right)^k$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^{10}}$.

SOLUTION. (a)

$$\begin{aligned}
\rho &= \lim_{k\to\infty} \sqrt[k]{a_k} \\
&= \lim_{k\to\infty} a_k^{1/k} \\
&= \lim_{k\to\infty} \left[\left(\frac{4k^2 - 3}{7k^2 + 6}\right)^k\right]^{1/k} \\
&= \lim_{k\to\infty} \frac{4k^2 - 3}{7k^2 + 6} \\
&= \lim_{k\to\infty} \frac{4 - 3/k^2}{7 + 6/k^2} \\
&= \frac{4}{7}.
\end{aligned}$$

Because $0 \le \rho < 1$, the series converges by the Root Test.

(b)

$$\begin{aligned}
\rho &= \lim_{k\to\infty} \sqrt[k]{a_k} \\
&= \lim_{k\to\infty} a_k^{1/k} \\
&= \lim_{k\to\infty} \left(\frac{2^k}{k^{10}}\right)^{1/k} \\
&= \lim_{k\to\infty} \frac{2}{k^{10/k}}.
\end{aligned}$$

Next, let us calculate $\lim_{k\to\infty} k^{10/k}$,

$$\begin{aligned}
\lim_{k\to\infty} k^{10/k} &= \lim_{k\to\infty} e^{\frac{10}{k}\ln k} \\
&= e^{\lim_{k\to\infty} \frac{10}{k}\ln k} \\
&= e^{\lim_{k\to\infty} \frac{\ln k}{\frac{k}{10}}} && \left[\lim_{k\to\infty} \frac{\ln k}{\frac{k}{10}} = \lim_{x\to\infty} \frac{\ln x}{\frac{x}{10}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{10}}\right] \\
&= e^0 && \text{[L'Hôpital's Rule]} \\
&= 1.
\end{aligned}$$

Therefore $\rho = \frac{2}{1} = 2 > 1$, the series diverges by the Root Test.

$\square$