MATH 2205 - Calculus II Lecture Notes 18

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1 Sequences and Series

1.1 Review of Sequences and Infinite Series

Theorem 1.1 (Limits of Sequences from Limits of Functions)**.** Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

Theorem 1.2 (Limit Laws for Sequences)**.** Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

(a) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$.
(b) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$, where c is a real number.
(c) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = AB$.
(d) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{A}{B}$ provided $B \neq 0$.

Theorem 1.3 (Squeeze Theorem for Sequences)**.** Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example 1.1 (Squeeze Theorem)**.** Find the limit of the sequence $b_n = \dfrac{\cos n}{n^2+1}$.

Solution. Note that $-1 \leq \cos n \leq 1$ for all n. It follows that

$$-\frac{1}{n^2+1} \leq \frac{\cos n}{n^2+1} \leq \frac{1}{n^2+1}.$$

Let $a_n = -\dfrac{1}{n^2+1}$ and $c_n = \dfrac{1}{n^2+1}$, we have $a_n \leq b_n \leq c_n$ for $n \geq 1$. Furthermore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$. By Theorem 1.3, we have $\lim_{n\to\infty} b_n = 0$. □

Theorem 1.4 (Bounded Monotonic Sequences)**.** A bounded monotonic sequence converges.

Definition 1.1 (Infinite series)**.** Given a sequence $\{a_1, a_2, a_3, \ldots,\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an *infinite series.* The *sequence of partial sums* $\{S_n\}$ associated with this series has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \text{for } n = 1, 2, 3, \ldots \end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series *converges* to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also *diverges.*

Definition 1.2 (Geometric Sequences)**.** A sequence has the form $\{r^n\}$ or $\{ar^n\}$, where the ratio r, a are real numbers, is called a geometric sequence.

Theorem 1.5 (Geometric Series)**.** Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges. More generally, if $|r| < 1$, then

$$\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}.$$

Example 1.2 (Telescoping series)**.** Evaluate the following series.

(a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{3^k} - \frac{1}{3^{k+1}} \right)$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

Solution.

(a)

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)\\
&\quad=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)\\
&\quad=\lim_{n\to\infty}\left[\left.\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)\right|_{k=1}+\left.\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)\right|_{k=2}+\left.\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)\right|_{k=3}+\cdots+\left.\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)\right|_{k=n}\right]\\
&\quad=\lim_{n\to\infty}\left[\left(\frac{1}{3^1}-\frac{1}{3^2}\right)+\left(\frac{1}{3^2}-\frac{1}{3^3}\right)+\left(\frac{1}{3^3}-\frac{1}{3^4}\right)+\cdots+\left(\frac{1}{3^n}-\frac{1}{3^{n+1}}\right)\right]\\
&\quad=\lim_{n\to\infty}\left[\frac{1}{3}+\left(-\frac{1}{3^2}+\frac{1}{3^2}\right)+\left(-\frac{1}{3^3}+\frac{1}{3^3}\right)+\cdots+\left(-\frac{1}{3^n}+\frac{1}{3^n}\right)-\frac{1}{3^{n+1}}\right]\\
&\quad=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{3^{n+1}}\right)\\
&\quad=\frac{1}{3}.
\end{aligned}$$

(b)

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\\
&\quad=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}\\
&\quad=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\
&\quad=\lim_{n\to\infty}\left[\left.\left(\frac{1}{k}-\frac{1}{k+1}\right)\right|_{k=1}+\left.\left(\frac{1}{k}-\frac{1}{k+1}\right)\right|_{k=2}+\left.\left(\frac{1}{k}-\frac{1}{k+1}\right)\right|_{k=3}+\cdots+\left.\left(\frac{1}{k}-\frac{1}{k+1}\right)\right|_{k=n}\right]\\
&\quad=\lim_{n\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]\\
&\quad=\lim_{n\to\infty}\left[\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}\right]\\
&\quad=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)\\
&\quad=\lim_{n\to\infty}\frac{n}{n+1}\\
&\quad=\lim_{n\to\infty}\frac{1}{1+1/n}\\
&\quad=1.
\end{aligned}$$

□

1.2 The Divergence and Integral Tests

Theorem 1.6 (Contrapositive and Converse)**.** If the statement "if p, then q" (i.e., $p \implies q$) is true, then its *contrapositive*, "if (not q), then (not p)" (i.e., $\neg q \implies \neg p$), is also true. However its *converse*, "if q, then p" (i.e., $q \implies p$), is not necessary true. In short,

$$p \implies q \equiv \neg q \implies \neg p,$$
$$p \implies q \not\equiv q \implies p,$$

where $A \equiv B$ means A and B are equivalent.

Example 1.3. Assume that Laramie is one of the cities in Wyoming, and both Laramie and Wyoming are unique in our universe.
Statement: If I live in Laramie, then I live in Wyoming. (true)
Contrapositive: If I don't live in Wyoming, then I don't live in Laramie. (true)
Converse: If I live in Wyoming, then I live in Laramie. (false)

Theorem 1.7 (Divergence Test)**.** If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Note: The converse of the above statement, if $\lim_{k\to\infty} a_k = 0$, then the series converges, might not be true.

Example 1.4 (Using the Divergence Test)**.** Determine whether the following series diverge or state that the Divergence Test is inconlusive.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$.
(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$.
(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.
(d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Solution.

(a) $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+1/k} = 1 \neq 0$. By Divergence Test, the series diverges.
(b) $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1+3^k}{2^k} = \lim_{k\to\infty} \frac{1/2^k + 3^k/2^k}{1} = \lim_{k\to\infty} \frac{1}{2^k} + \left(\frac{3}{2}\right)^k = \infty \neq 0$. By Divergence Test, the series diverges.
(c) $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$. In this case, the terms of the series approach zero, so the Divergence Test is inconclusive. Remember, the Divergence Test cannot be used to prove that a series converges.
(d) $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^2} = 0$. Again, the Divergecen Test is inconclusive.

□

Theorem 1.8 (Harmonic Series)**.** The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges – even though the terms of the series approach zero.

Proposition 1.1.

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \approx \ln n + \gamma,$$

where $\gamma \approx 0.57721\ldots$.

Theorem 1.9 (Integral Test)**.** Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

Example 1.5 (Applying the Integral Test)**.** Determine whether the following series converge.

$$\sum_{k=1}^{\infty} \frac{k}{k^2+1}.$$

Solution. Given that $a_k = \dfrac{k}{k^2+1}$, and

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k^2+1} = \lim_{k\to\infty} \frac{\frac{k}{k}}{\frac{k^2}{k} + \frac{1}{k}} = \lim_{k\to\infty} \frac{1}{k + 1/k} = 0.$$

Then the Divergence Test is inconclusive. Now let us use Integral Test, the function associated with this series is $f(x) = \dfrac{x}{x^2+1}$, which is positive, decreasing for $x \geq 1$. The integral that determines convergence is

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^2+1}\, dx &= \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\, dx \\
&= \lim_{b\to} \frac{1}{2} \int_{u(1)}^{u(b)} \frac{1}{u}\, du && [u = x^2+1, du = 2x\, dx] \\
&= \frac{1}{2} \lim_{b\to\infty} \ln|u| \Big|_{u(1)=2}^{u(b)=b^2+1} \\
&= \frac{1}{2} \lim_{b\to\infty} [\ln(b^2+1) - \ln 2] \\
&= \infty.
\end{aligned}$$

Because the integal diverges, the series diverges. □