MATH 2205 - Calculus II Lecture Notes 16

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1 Differential Equations and Sequences

1.1 Review of Separable First-Order Differential Equations, Sequences

Definition 1.1 (Separable First-Order Differential Equations)**.** If the first-order differential equation can be written in the form $g(y)y'(t) = h(t)$, in which the terms that involve y appear on one side of the equation *separated* from the terms that involve t, is said to be *separable*. We can solve the equaiton by integrating both sides of the equation with respect to t:

$$\int g(y)y'(t)\,dt = \int h(t)\,dt \implies \int g(y)\,dy = \int h(t)\,dt.$$

Definition 1.2 (Sequence)**.** A *sequence* $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a *recurrence relation* of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an *explicit formula* of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Definition 1.3 (Limit of a Sequence)**.** If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases – that is, if a_n can be made arbitrarily close to L by taking n sufficiently large – then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence *converges* to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence *diverges*.

Example 1.1 (Limits of Sequences)**.** Write the first four terms of each sequence. If you believe the sequence converges, make a conjecture about its limit. If the sequence appears to diverge, explain why.

(a) $\left\{\dfrac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$.

(b) $\{\cos n\pi\}_{n=1}^{\infty}$.

(c) $\{a_n\}_{n=1}^{\infty}$, where $a_{n+1} = -2a_n, a_1 = 1$.

Solution.

(a) The first four terms of the sequence are

$$\left\{\frac{(-1)^1}{1^2+1}, \frac{(-1)^2}{2^2+1}, \frac{(-1)^3}{3^2+1}, \frac{(-1)^4}{4^2+1}, \ldots\right\} = \left\{-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, \ldots\right\}$$

Observe that $-\dfrac{1}{n^2} < \dfrac{(-1)^n}{n^2+1} < \dfrac{1}{n^2}$, and

$$\lim_{n\to\infty} -\frac{1}{n^2} = 0 \text{ and } \lim_{n\to\infty} \frac{1}{n^2} = 0 \implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0.$$

We can also graph the sequence (see Figure 1).

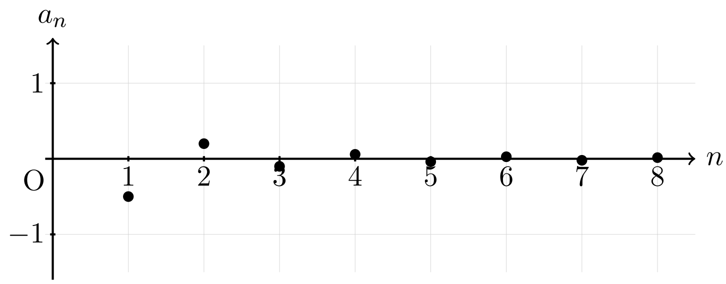

Figure 1: $\left\{\dfrac{(-1)^n}{n^2+1}\right\}_{n=1}^{8}$

(b) The first four terms of the sequence are

$$\{\cos 1\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \ldots\} = \{-1, 1, -1, 1, \ldots\}.$$

In this case, the terms of the sequence alternate between -1 and $+1$, and never approach a single value. Therefore, the sequence diverges. We can also graph the sequence (see Figure 2).

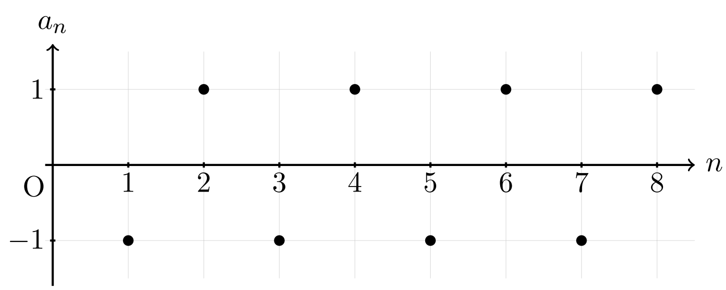

Figure 2: $\{\cos n\pi\}_{n=1}^{8}$

(c) First let us derive the explicit formula for the sequence using the recurrence relation $a_{n+1} = -2a_n, a_1 = 1$, apply the relation to a_n $(n-1)$ times we have

$$a_n = -2a_{n-1} = -2(-2a_{n-2}) = (-2)^2 a_{n-2} = (-2)^3 a_{n-3} = \cdots = (-2)^{n-1} a_1 = (-2)^{n-1}.$$

Then the first four terms of the sequence are

$$\begin{aligned}\{a_1, a_2, a_3, a_4, \ldots\} &= \{(-2)^{1-1}, (-2)^{2-1}, (-2)^{3-1}, (-2)^{4-1}, \ldots\}\\ &= \{(-2)^0, (-2)^1, (-2)^2, (-2)^3, \ldots\}\\ &= \{1, -2, 4, -8, \ldots\}.\end{aligned}$$

The magnitude of the terms increase without bound, the sequence thus diverges.

□

1.2 Sequences

Theorem 1.1 (Limits of Sequences from Limits of Functions)**.** Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

Theorem 1.2 (Limit Laws for Sequences)**.** Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

(a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = A \pm B$.

(b) $\lim\limits_{n\to\infty} ca_n = cA$, where c is a real number.

(c) $\lim\limits_{n\to\infty} a_n b_n = AB$.

(d) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ provided $B \neq 0$.

Example 1.2 (Limits of sequences)**.** Determine the limits of the following sequences.

(a) $a_n = \dfrac{3n^3}{n^3+1}$.

(b) $b_n = \left(\dfrac{n+5}{n}\right)^n$.

(c) $c_n = n^{1/n}$.

Solution.

(a)

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3n^3}{n^3+1} \\
&= \lim_{n\to\infty} \frac{3}{1+1/n^3} && \text{[divide both top and bottom by } n^3\text{]} \\
&= \frac{\lim\limits_{n\to\infty} 3}{\lim\limits_{n\to\infty}(1+1/n^3)} \\
&= \frac{3}{1} \\
&= 3.
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \left(\frac{n+5}{n}\right)^n \\
&= \lim_{n\to\infty} e^{\ln\left(\frac{n+5}{n}\right)^n} && [a = e^{\ln a}] \\
&= e^{\lim\limits_{n\to\infty} n\ln(1+5/n)} && [\ln a^b = b\ln a]
\end{aligned}$$

Next, let's calculate $\lim\limits_{n\to\infty} n\ln(1+5/n)$,

$$\begin{aligned}
\lim_{n\to\infty} n\ln(1+5/n) &= \lim_{n\to\infty} \frac{\ln(1+5/n)}{1/n} \\
&= \lim_{x\to\infty} \frac{\ln(1+5/x)}{1/x} && \text{[By Theorem 1.1, Indeterminate form 0/0]} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+5/x}\left(-\frac{5}{x^2}\right)}{-\frac{1}{x^2}} && \text{[L'Hôpital's Rule]} \\
&= 5\lim_{x\to\infty} \frac{1}{1+5/x} \\
&= 5.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} b_n = e^5$.

(c)

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} n^{1/n} \\
&= \lim_{n\to\infty} e^{\ln n^{1/n}} && [a = e^{\ln a}] \\
&= \lim_{n\to\infty} e^{\frac{1}{n}\ln n} && [\ln a^b = b\ln a] \\
&= e^{\lim\limits_{n\to\infty} \frac{1}{n}\ln n}.
\end{aligned}$$

Then let us compute $\lim_{n\to\infty} \frac{1}{n}\ln n$ as follows

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\ln n &= \lim_{n\to\infty} \frac{1}{x}\ln x && [\text{By Theorem 1.1}] \\
&= \lim_{n\to\infty} \frac{\ln x}{x} && [\text{Indeterminate form } \infty/\infty] \\
&= \lim_{n\to\infty} \frac{\frac{1}{x}}{1} && [\text{L'Hôpital's Rule}] \\
&= \lim_{n\to\infty} \frac{1}{x} \\
&= 0.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} c_n = e^0 = 1$.

□

Definition 1.4 (Terminology for Sequences)**.**

(a) $\{a_n\}$ is *increasing* if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$.
(b) $\{a_n\}$ is *nondecreasing* if $a_{n+1} \geq a_n$; for example, $\{0, 1, 1, 1, 2, 2, 3, \ldots\}$.
(c) $\{a_n\}$ is *decreasing* if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -2, \ldots\}$.
(d) $\{a_n\}$ is *nonincreasing* if $a_{n+1} \leq a_n$; for example, $\{2, 1, 1, 0, -2, -2, -3, \ldots\}$.
(e) $\{a_n\}$ is *monotonic* if it is either nonincreasing or nondecreasing (it moves in one direction).
(f) $\{a_n\}$ is *bounded* if there is number M such that $|a_n| \leq M$, for all relevant values of n.

Theorem 1.3 (Squeeze Theorem for Sequences)**.** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Example 1.3 (Squeeze Theorem)**.** Find the limit of the sequence $b_n = \dfrac{\cos n}{n^2+1}$.

Solution. Note that $-1 \leq \cos n \leq 1$ for all n. It follows that

$$-\frac{1}{n^2+1} \leq \frac{\cos n}{n^2+1} \leq \frac{1}{n^2+1}.$$

Let $a_n = -\dfrac{1}{n^2+1}$ and $c_n = \dfrac{1}{n^2+1}$, we have $a_n \leq b_n \leq c_n$ for $n \geq 1$. Furthermore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$. By Theorem 1.3, we have $\lim_{n\to\infty} b_n = 0$. □

Theorem 1.4 (Bounded Monotonic Sequences)**.** *A bounded monotonic sequence converges.*

Theorem 1.5 (Growth Rates of Sequences)**.** The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s and $b > 1$.

1.3 Infinite Series

Definition 1.5 (Infinite series)**.** Given a sequence $\{a_1, a_2, a_3, \dots, \}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an *infinite series*. The *sequence of partial sums* $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \text{for } n = 1, 2, 3, \dots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series *converges* to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also *diverges*.

Definition 1.6 (Geometric Sequences)**.** A sequence has the form $\{r^n\}$ or $\{ar^n\}$, where the ratio r, a are real numbers, is called a geometric sequence.

Theorem 1.6 (Geometric Sequences)**.** Let r be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1, \\ 1 & \text{if } r = 1, \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.

Theorem 1.7 (Geometric Series)**.** Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \ge 1$, then the series diverges.

Proof. Assume that $|r|< 1$, then consider the partial sum $S_n = \sum_{k=m}^{n} ar^k$, where $m \leq n$ then

$$\begin{aligned}
S_n - rS_n &= (1-r)S_n \\
&= \sum_{k=m}^{n} ar^k - r\sum_{k=m}^{n} ar^k \\
&= \sum_{k=m}^{n} ar^k - \sum_{k=m}^{n} arr^k \\
&= \sum_{k=m}^{n} ar^k - \sum_{k=m}^{n} ar^{k+1} \\
&= \sum_{k=m}^{n} ar^k - \sum_{k=m+1}^{n+1} ar^k \\
&= \left(ar^m + \sum_{k=m+1}^{n} ar^k\right) - \left(\sum_{k=m+1}^{n} ar^k + ar^{n+1}\right) \\
&= ar^m + \sum_{k=m+1}^{n} ar^k - \sum_{k=m+1}^{n} ar^k - ar^{n+1} \\
&= ar^m - ar^{n+1}.
\end{aligned}$$

Then solve for S_n, we have

$$S_n = \frac{ar^m - ar^{n+1}}{1-r}.$$

Then the geometric series becomes

$$\begin{aligned}
\sum_{k=m}^{\infty} ar^k &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} \frac{ar^m - ar^{n+1}}{1-r} \\
&= \frac{ar^m}{1-r}.
\end{aligned}$$

Let $m = 0$, then we have

$$\sum_{k=0}^{\infty} ar^k = \left.\frac{ar^m}{1-r}\right|_{m=0} = \frac{a}{1-r}.$$

$\square$

Example 1.4. Evaluate the following geometric series or state that the series diverges.

(a) $\displaystyle\sum_{k=0}^{\infty} 1.1^k$.

(b) $\displaystyle\sum_{k=0}^{\infty} e^{-k}$.

(c) $\displaystyle\sum_{k=2}^{\infty} 3(-0.75)^k$.

Solution.

(a) The ratio of this geometric series is $r = 1.1$. Because $|r| \geq 1$, the series diverges.

(b) Note that $e^{-k} = (e^{-1})^k = \left(\frac{1}{e}\right)^k$. The ratio of this geometric series is $r = \frac{1}{e} < 1$, and its first term $a = 1$. Because $|r| < 1$, the series converges and its value is

$$\sum_{k=0}^{\infty} e^{-k} = \frac{a}{1-r} = \frac{1}{1-1/e} = \frac{e}{e-1}.$$

(c) Note that $a = 3$, $r = -0.75$ and $|r| = 0.75 < 1$, then the geometric series converges. Therefore,

$$\begin{aligned}\sum_{k=2}^{\infty} 3(-0.75)^k &= \frac{ar^2}{1-r} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}] \\ &= \frac{3(-0.75)^2}{1-(-0.75)} \\ &= \frac{27/16}{7/4} \\ &= \frac{27}{28}.\end{aligned}$$

□

Example 1.5 (Decimal expansion as geometric series)**.** Write $1.0\overline{35} = 1.035353535\ldots$ as a geometric series and express its value as a fraction.

Solution.

$$\begin{aligned}1.0\overline{35} &= 1.035353535\ldots \\ &= 1 + 0.035 + 0.00035 + 0.0000035 + 0.000000035 + \cdots \\ &= 1 + \sum_{k=0}^{\infty} 0.035 \cdot 0.01^k && [a = 0.035, r = 0.01, |r| < 1] \\ &= 1 + \frac{0.035}{1-0.01} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}] \\ &= 1 + \frac{35}{1000-10} \\ &= 1 + \frac{35}{990} \\ &= \frac{990+35}{990} \\ &= \frac{1025}{990} \\ &= \frac{205}{198}.\end{aligned}$$

□