MATH 2205 - Calculus II Lecture Notes 13

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1 Integration Techniques

1.1 Review of Numerical Integration

Definition 1.1 (Absolute and Relative Error). Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common meaasures of the error in c as an approximation to x:

$$\textbf{absolute error} = |c - x|$$

and

$$\textbf{relative error} = \frac{c - x}{x}, \text{(if } x \neq 0).$$

Definition 1.2. Suppose f is defined an integrable on $[a, b]$. The *Midpoint Rule approximation* to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x = \sum_{k=1}^{n} f\left(\frac{x_{n-1} + x_n}{2}\right)\Delta x,$$

where $\Delta x = (b - a)/n$, $x_0 = a$, $x_k = a + k\Delta x$, and $m_k = (x_{k-1} + x_k)/2 = a + (k - 1/2)\Delta x$ is the midpoint of $[x_k, x_k]$, for $k = 1, \ldots, n$.

Definition 1.3 (Trapezoid Rule). Suppose f is defined and integrable on $[a, b]$. The *Trapezoid Rule approximation* to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x.$$

where $\Delta x = (b - a)/n$ and $x_k = a + k\Delta x$, for $k = 0, 1, 2, \ldots, n$.

Definition 1.4 (Simpson's Rule). Suppose f is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The *Simpson's Rule approximation* to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$\begin{aligned} S(n) &= [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]\frac{\Delta x}{3} \\ &= \sum_{k=0}^{n/2-1} [f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})]\frac{\Delta x}{3}. \end{aligned}$$

where n is an even integer, $\Delta x = (b - a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

1.2 Improper Integrals

Definition 1.5 (Improper Integrals over Infinite Intervals)**.**

(a) If f is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

(b) If f is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

(c) If f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx,$$

where c is any real number.

If the limits in the above cases exist, then the improper integrals *converge*; otherwise, they *diverge*.

Example 1.1 (Infinite intervals)**.** Evaluate each integral.

(a) $\displaystyle\int_0^\infty e^{-3x}\,dx.$

(b) $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}.$

Solution.

(a)

$$\begin{aligned}
\int_0^\infty e^{-3x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-3x}\,dx\\
&= \lim_{b\to\infty} -\frac{1}{3}\int_0^b e^{\overbrace{-3x}^{u}}\cdot\underbrace{(-3)\,dx}_{du}\\
&= \lim_{b\to\infty} -\frac{1}{3}\int_{u(0)}^{u(b)} e^u\,du\\
&= \lim_{b\to\infty} -\frac{1}{3}e^u\Big|_{u(0)=0}^{u(b)=-3b}\\
&= \lim_{b\to\infty} -\frac{1}{3}(e^{-3b}-e^0)\\
&= -\frac{1}{3}(\lim_{b\to\infty} e^{-3b}-1)\\
&= -\frac{1}{3}(0-1)\\
&= \frac{1}{3}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}\, dx + \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \arctan x \Big|_a^0 + \lim_{b\to\infty} \arctan x \Big|_0^b \\
&= \lim_{a\to-\infty} (\arctan 0 - \arctan a) + \lim_{b\to\infty} (\arctan b - \arctan 0) \\
&= (0 - \lim_{a\to-\infty} \arctan a) + (\lim_{b\to\infty} \arctan b - 0) \\
&= -\lim_{a\to-\infty} \arctan a + \lim_{b\to\infty} \arctan b \\
&= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} \\
&= \pi.
\end{aligned}$$

□

Example 1.2 (The family $f(x) = 1/x^p$)**.** Consider the family of functions $f(x) = 1/x^p$, where p is a real number. For what values of p does $\displaystyle\int_1^{\infty} f(x)\, dx$ converge?

Solution. Note that

$$\int f(x)\, dx = \int \frac{1}{x^p}\, dx = \int x^{-p}\, dx = \begin{cases} \dfrac{x^{-p+1}}{-p+1} + C & \text{if } p \neq 1, \\ \ln|x| + C & \text{if } p = 1. \end{cases}$$

Therefore, when $p = 1$,

$$\int_1^{\infty} f(x)\, dx = \int_1^{\infty} x^{-p}\, dx = \lim_{b\to\infty} \int_1^b x^{-1}\, dx = \lim_{b\to\infty} \ln|x| \Big|_1^b = \lim_{b\to\infty} (\ln|b| - \ln 1) = \lim_{b\to\infty} \ln b = \infty.$$

So the integral diverges when $p = 1$. Next, let us consider the case $p \neq 1$.

$$\int_1^{\infty} f(x)\, dx = \int_1^{\infty} x^{-p}\, dx = \lim_{b\to\infty} \int_1^b x^{-p}\, dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1} \Big|_1^b = \lim_{b\to\infty} \frac{1}{-p+1} \left(b^{-p+1} - 1\right).$$

Hence we need to consider the cases when $p > 1$ and $p < 1$.

- If $p > 1$, then $-p+1 = 1-p < 0$, then $b^{-p+1} = b^{1-p} = \dfrac{1}{b^{p-1}}$. Therefore,

  $$\int_1^{\infty} f(x)\, dx = \lim_{b\to\infty} \frac{1}{-p+1} \left(b^{-p+1} - 1\right) = \frac{1}{-p+1} \left(\lim_{b\to\infty} \frac{1}{b^{p-1}} - 1\right) = \frac{1}{-p+1} (0-1) = \frac{1}{p-1}.$$

  That is, when $p > 1$, the integral converges to $\frac{1}{p-1}$.
- If $p < 1$, then $-p+1 = 1-p > 0$. So we can obtain

  $$\int_1^{\infty} f(x)\, dx = \lim_{b\to\infty} \frac{1}{-p+1} \left(b^{-p+1} - 1\right) = \frac{1}{-p+1} \left(\lim_{b\to\infty} b^{1-p} - 1\right) = \infty.$$

  In other words, when $p < 1$, the integral diverges.

In a nutshell, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$, if $p > 1$, it diverges otherwise. $\square$

Definition 1.6 (Improper Integrals with an Unbounded Integrand)**.**

(a) Suppose f is continous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

(b) Suppose f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} f(x)\,dx.$$

(c) Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \lim_{c\to p^-} \int_a^c f(x)\,dx + \lim_{d\to p^+} \int_d^b f(x)\,dx.$$

If the limits in above cases exist, then the improper integrals *converge*; otherwise, they *diverge*.

Example 1.3 (Infinite integrand)**.** Find the area of the region R between the graph of $f(x) = \dfrac{1}{\sqrt{9-x^2}}$ and the x-axis on the interval $(-3, 3)$ (if it exists).

Solution.

$$\begin{aligned}
A = \int_{-3}^3 f(x)\,dx &= \int_{-3}^3 \frac{1}{\sqrt{9-x^2}}\,dx \\
&= 2\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx && [\tfrac{1}{\sqrt{9-x^2}} \text{ is even}] \\
&= 2\lim_{b\to 3^-}\int_0^b \frac{1}{\sqrt{9-x^2}}\,dx && [f(x) \text{ is unbounded at } x = 3] \\
&= 2\lim_{b\to 3^-}\int_0^b \frac{1}{\sqrt{1-(x/3)^2}}\frac{1}{3}\,dx \\
&= 2\lim_{b\to 3^-}\int_{u(0)}^{u(b)} \frac{1}{\sqrt{1-u^2}}\,du && [u = \tfrac{3}{x}, du = \tfrac{1}{3}dx] \\
&= 2\lim_{b\to 3^-} \arcsin u \Big|_{u(0)=0}^{u(b)=\frac{b}{3}} \\
&= 2\left(\lim_{b\to 3^-} \arcsin\frac{b}{3} - \arcsin 0\right) && [\lim_{b\to 3^-} \arcsin\tfrac{b}{3} = \arcsin 1 = \tfrac{\pi}{2}] \\
&= 2\left(\frac{\pi}{2} - 0\right) \\
&= \pi.
\end{aligned}$$

$\square$

Example 1.4 (Infinite integrand at an interior point)**.** Evaluate $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$.

Solution. Note that the integrand is unbounded at $x = 2$ which is an interior point of the interval of integration. We split the interval into two subintervals and evaluate an improper integral on each subinterval:

$$\begin{aligned}
\int_1^{10} \frac{dx}{(x-2)^{1/3}} &= \lim_{c\to 2^-} \int_1^c (x-2)^{-1/3}\,dx + \lim_{d\to 2^+} \int_d^{10} (x-2)^{-1/3}\,dx \\
&= \lim_{c\to 2^-} \frac{3}{2}(x-2)^{2/3}\Big|_1^c + \lim_{d\to 2^+} (x-2)^{2/3}\Big|_d^{10} \\
&= \lim_{c\to 2^-} \frac{3}{2}[(c-2)^{2/3} - (-1)^{2/3}] + \lim_{d\to 2^+} \frac{3}{2}[8^{2/3} - (d-2)^{2/3}] \\
&= \frac{3}{2}[\lim_{c\to 2^-} (c-2)^{2/3} - 1] + \frac{3}{2}[4 - \lim_{d\to 2^+} (d-2)^{2/3}] \\
&= \frac{3}{2}[(0-1) + (4-0)] \\
&= \frac{9}{2}.
\end{aligned}$$

$\square$