MATH 2205 - Calculus II Lecture Notes 11

Last update: June 20, 2019

1 Integration Techniques

1.1 Review for Trigonometric Integrals and Trigonometric Substitution

1.1.1 Integrating Powers of $\sin x$ or $\cos x$

Procedure 1.1. Strategies for evaluating integrals of the form $\displaystyle\int \sin^m x\,dx$ or $\displaystyle\int \cos^n x\,dx$, where m and n are positive integers, using trigonometric identities.

(a) Integrals involving odd powers of $\cos x$ (or $\sin x$) are most easily evaluated by splitting off a single factor of $\cos x$ (or $\sin x$). For example, rewrite $\cos^5 x$ as $\cos^4 x \cdot \cos x$.

(b) With even positive powers of $\sin x$ or $\cos x$, we use the half-angle formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } \cos^2\theta = \frac{1+\cos 2\theta}{2},$$

to reduce the powers in the integrand.

1.1.2 Integrating Products of Powers of $\sin x$ and $\cos x$

Procedure 1.2. Strategies for evaluating integrals of the form $\displaystyle\int \sin^m x\cos^n x\,dx$.

(a) When m is odd and positive, n real. Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.

(b) When n is odd and positive, m real. Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

(c) When m, n are both even and nonnegative. Use half-angle formulas to transform the integrand into polynomial in $\cos 2x$ and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

1.2 Trigonometric Substitutions

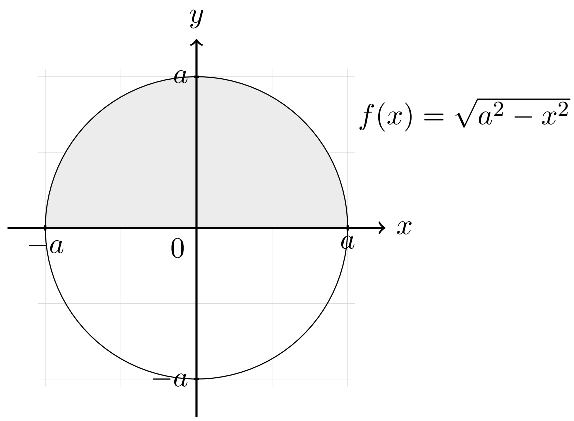
Procedure 1.3. Integrals Involving $a^2 - x^2$, $a^2 + x^2$ or $x^2 - a^2$

(a) The integral contains $a^2 - x^2$. Let $x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$ for $|x| \le a$. Then $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\cos^2\theta) = a^2\cos^2\theta$.

(b) The integral contains $a^2 + x^2$. Let $x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$.

(c) The integral contains $x^2 - a^2$. Let $x = a\sec\theta$, $\begin{cases} 0 \le \theta < \pi/2 & \text{for } x \ge a \\ \pi/2 < \theta \le \pi & \text{for } x \le -a \end{cases}$. Then $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$.

Example 1.1 (Area of a circle)**.** Verify that the area of a circle of radius a is πa^2.



Solution. Solve for y in the circle equation as follows

$$x^2 + y^2 = a^2 \implies y = \sqrt{a^2 - x^2},$$

which represents the top half of the circle. So the area of the circle twice the area of the region under the curve $y = \sqrt{a^2 - x^2}$. Let $f(x) = \sqrt{a^2 - x^2}$, then the area of the region under the curve is

$$\begin{aligned}
A &= 2\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx \\
&= 2 \times 2\int_{0}^{a} \sqrt{a^2 - x^2}\, dx && [x = a\sin\theta, dx = a\cos\theta\, d\theta, \theta \in [0, \pi/2]] \\
&= 4\int_{0}^{\pi/2} \sqrt{a^2 - (a\sin\theta)^2} a\cos\theta\, d\theta \\
&= 4\int_{0}^{\pi/2} a\cos\theta a\cos\theta\, d\theta \\
&= 4a^2\int_{0}^{\pi/2} \cos^2\theta\, d\theta \\
&= 2a^2\int_{0}^{\pi/2} \cos 2\theta + 1\, d\theta \\
&= 2a^2\left(\int_{0}^{\pi/2} \cos 2\theta\, d\theta + \int_{0}^{\pi/2} d\theta\right) \\
&= 2a^2\left(\frac{1}{2}\int_{0}^{\pi/2} \cos \underbrace{2\theta}_{u} \underbrace{2\, d\theta}_{du} + \int_{0}^{\pi/2} d\theta\right) \\
&= 2a^2\left(\frac{1}{2}\int_{u(0)}^{u(\pi/2)} \cos u\, du + \int_{0}^{\pi/2} d\theta\right) \\
&= 2a^2\left(\frac{1}{2}\sin u\Big|_{0}^{\pi} + \theta\Big|_{0}^{\pi/2}\right) \\
&= 2a^2(0 + \frac{\pi}{2}) \\
&= \pi a^2.
\end{aligned}$$

□

1.3 Partial Fractions

Procedure 1.4 (Partial Fractions with Simple Linear Factors)**.** Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

(a) *Factor the denominator* q in the form $(x-r_1)(x-r_2)\cdots(x-r_n)$, where $r_1, \ldots, r_n$ are real numbers.

(b) *Partial fraction decomposition.* Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \cdots + \frac{A_n}{(x-r_n)}.$$

(c) *Clear denominators.* Multiply both sides of the equation in Step (b) by $q(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$, which produces conditions for $A_1, \ldots, A_n$.

(d) *Solve for coefficients.* Equate like powers of x in Step (c) to solve for the undetermined coefficients $A_1, \ldots, A_n$.

Example 1.2 (Integrating with partial fractions)**.**

(a) Find the partial fraction decomposition for $f(x) = \dfrac{3x^2+7x-2}{x^3-x^2-2x}$.

(b) Evaluate $\displaystyle\int f(x)\,dx$.

SOLUTION.

(a) We want to decompose $f(x)$ as follows

$$\begin{aligned} f(x) = \frac{3x^2+7x-2}{x^3-x^2-2x} &= \frac{3x^2+7x-2}{x(x^2-x-2)} \\ &= \frac{3x^2+7x-2}{x(x-2)(x+1)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}. \end{aligned} \tag{1.1}$$

Now multiply both sides of (1.1) by $x(x-2)(x+1)$, we have

$$3x^2 + 7x - 2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2). \tag{1.2}$$

Then we use shortcut to determine A, B, C. Substituting $x = 0, 2, -1$ in (1.2) allows us to solve directly for the coefficients:

$$\begin{aligned} &\text{Letting } x = 0 : -2 = A\cdot(-2)\cdot 1 + B\cdot 0 + C\cdot 0 \implies A = 1; \\ &\text{Letting } x = 2 : 24 = A\cdot 0 + B\cdot 6 + C\cdot 0 \implies B = 4; \\ &\text{Letting } x = -1 : -6 = A\cdot 0 + B\cdot 0 + C\cdot 3 \implies B = -2. \end{aligned}$$

Then the partial fraction decomposition is

$$f(x) = \frac{1}{x} + \frac{4}{x-2} - \frac{2}{x+1}.$$

(b)

$$\begin{aligned}\int f(x)\,dx &= \int \frac{1}{x} + \frac{4}{x-2} - \frac{2}{x+1}\,dx \\ &= \int \frac{1}{x}\,dx + 4\int \frac{1}{x-2} - 2\int \frac{1}{x+1}\,dx \\ &= \ln|x| + 4\ln|x-2| - 2\ln|x+1| + D,\end{aligned}$$

where D is a constant.

□

Procedure 1.5 (Partial Fractions for Repeated Linear Factors)**.** Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m},$$

where $A_1, \ldots, A_m$ are constants to be determined.

Example 1.3 (Integrating with repeated linear factors)**.** Evalute $\int f(x)\,dx$, where $f(x) = \dfrac{5x^2-3x+2}{x^3-2x^2}$.

Solution. First let us decompose $f(x)$ as follows

$$\begin{aligned}f(x) = \frac{5x^2-3x+2}{x^3-2x^2} &= \frac{5x^2-3x+2}{x^2(x-2)} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}.\end{aligned}$$

Multiplying both sides by $x^2(x-2)$ gives

$$5x^2 - 3x + 2 = Ax(x-2) + B(x-2) + Cx^2 = (A+C)x^2 + (B-2A)x - 2B.$$

Next equate like powers of x to solve for undetermined A, B and C, we get a system of equations as below:

$$\begin{cases} A+C &= 5 \\ B-2A &= -3 \\ -2B &= 2 \end{cases} \implies \begin{cases} A &= 1 \\ B &= -1 \\ C &= 4 \end{cases} \tag{1.3}$$

Then the partial fraction decomposition of $f(x)$ is

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}.$$

Then integrating $f(x)$ yields

$$\begin{aligned}\int f(x)\,dx = \int \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}\,dx &= \int \frac{1}{x}\,dx - \int x^{-2}\,dx + 4\frac{1}{x-2}\,dx \\ &= \ln|x| + \frac{1}{x} + 4\ln|x-2| + D,\end{aligned}$$

where D is a constant. □

Procedure 1.6 (Partial Fractions with Simple Irreducible Quadratic Factors)**.** Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are unknown coefficients to be determined.

Example 1.4. Give the appropriate form of the partial fraction decompositin for the following functions.

(a) $\dfrac{x^2 + 1}{x^4 - 4x^3 - 32x^2}$.

(b) $\dfrac{10}{(x-2)^2(x^2 + 2x + 2)}$.

Solution.

(a)

$$\frac{x^2 + 1}{x^4 - 4x^3 - 32x^2} = \frac{x^2 + 1}{x^2(x^2 - 4x - 32)} = \frac{x^2 + 1}{x^2(x-8)(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-8} + \frac{D}{x+4}.$$

(b)

$$\frac{10}{(x-2)^2(x^2 + 2x + 2)} = \frac{10}{(x-2)^2(x^2 + 2x + 2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx + D}{x^2 + 2x + 2}.$$

□

Example 1.5 (Integrating with partial fraciotns)**.** Evaluate $\displaystyle\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)}\, dx$.

Solution. Observe that $x^2 - 2x + 3$ is irreducible, then we have the partial fraction decomposition of the integrand as follows,

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 3}.$$

Then multiplying both sides by $(x-2)(x^2 - 2x + 3)$ gives

$$\begin{aligned} 7x^2 - 13x + 13 &= A(x^2 - 2x + 3) + (Bx + C)(x - 2) \\ &= (A + B)x^2 + (-2A - 2B + C)x + (3A - 2C). \end{aligned}$$

Equating coefficients of equal powers of x results in the equations

$$\begin{cases} A + B &= 7 \\ -2A - 2B + C &= -13 \\ 3A - 2C &= 13 \end{cases} \Longrightarrow \begin{cases} A &= 5 \\ B &= 2 \\ C &= 1 \end{cases}$$

Then the integral becomes

$$\begin{aligned}
\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx &= \int \frac{5}{x-2} + \frac{2x+1}{x^2-2x+3}\,dx \\
&= 5\int \frac{1}{x-2}\,dx + \int \frac{(2x-2)+3}{x^2-2x+3}\,dx \\
&= 5\ln|x-2| + \int \frac{(2x-2)}{x^2-2x+3}\,dx + \int \frac{3}{(x^2-2x+1)+2}\,dx \\
&= 5\ln|x-2| + \int \frac{1}{\underbrace{x^2-2x+3}_{u}} \underbrace{(2x-2)\,dx}_{du} + 3\int \frac{1}{2+(x-1)^2}\,dx \\
&= 5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}} \int \frac{1}{1+[(x-1)/\sqrt{2}]^2}\frac{1}{\sqrt{2}}\,dx \\
&= 5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + D.
\end{aligned}$$

□

Proposition 1.1 (Partial Fraction Decomposition)**.** Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

(a) *Simple linear factor.* A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.

(b) *Repeated linear factor.* A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

(c) *Simple irreducible quadratic factor.* An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax+B}{ax^2+bx+c}.$$

(d) *Repeated irreducible quadratic factor.* An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$