MATH 2205 - Calculus II Lecture Notes 10

Last update: June 11, 2019

1 Integration Techniques

1.1 Review for Integration by Parts for Integrals

Theorem 1.1 (Integration by Parts)**.** Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v du.$$

Theorem 1.2 (Integration by Parts for Definite Integrals)**.** Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

1.2 Trigonometric Integrals

1.2.1 Integrating Powers of $\sin x$ or $\cos x$

Procedure 1.1. Strategies for evaluating integrals of the form $\displaystyle\int \sin^m x\,dx$ or $\displaystyle\int \cos^n x\,dx$, where m and n are positive integers, using trigonometric identities.

(a) Integrals involving odd powers of $\cos x$ (or $\sin x$) are most easily evaluated by splitting off a single factor of $\cos x$ (or $\sin x$). For example, rewrite $\cos^5 x$ as $\cos^4 x \cdot \cos x$.

(b) With even positive powers of $\sin x$ or $\cos x$, we use the half-angle formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } \cos^2\theta = \frac{1+\cos 2\theta}{2},$$

to reduce the powers in the integrand.

1.2.2 Integrating Products of Powers of $\sin x$ and $\cos x$

Procedure 1.2. Strategies for evaluating integrals of the form $\displaystyle\int \sin^m x \cos^n x\,dx$.

(a) When m is odd and positive, n real. Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.

(b) When n is odd and positive, m real. Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

(c) When m, n are both even and nonnegative. Use half-angle formulas to transform the integrand into polynomial in $\cos 2x$ and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Proposition 1.1 (Reduction Formulas)**.** Assume n is a positive integer.

(a) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

(b) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

(c) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, n \neq 1.$

(d) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, n \neq 1.$

Proof.

(a)

$$\begin{aligned}
\int \sin^n x\,dx &= \int \sin x \sin^{n-1} x\,dx \qquad [u = \sin^{n-1} x, dv = \sin x\,dx \implies v = -\cos x] \\
&= \sin^{n-1} x(-\cos x) - \int (-\cos x) d\sin^{n-1} x \qquad [\int udv = uv - \int vdu] \\
&= -\sin^{n-1} x\cos x + \int \cos x(n-1)\sin^{n-2} x\cos x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \cos^2 x\sin^{n-2} x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x - \sin^n x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.
\end{aligned}$$

Then solve for $\int \sin^n x\,dx$, we can obtain

$$\begin{aligned}
n\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx \\
\implies \int \sin^n x\,dx &= -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.
\end{aligned}$$

(b)

$$\begin{aligned}
\int \cos^n x\,dx &= \int \cos x\cos^{n-1} x\,dx \qquad [u = \cos^{n-1} x, dv = \cos x\,dx \implies v = \sin x] \\
&= \cos^{n-1} x\sin x - \int \sin x\,d\cos^{n-1} x \\
&= \cos^{n-1} x\sin x - \int \sin x(n-1)\cos^{n-2} x(-\sin x)\,dx \\
&= \cos^{n-1} x\sin x + (n-1)\int \sin^2 x\cos^{n-2} x\,dx \\
&= \cos^{n-1} x\sin x + (n-1)\int (1-\cos 2x)\cos^{n-2} x\,dx \\
&= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.
\end{aligned}$$

Similarly, solve for $\int \cos^n x\,dx$,

$$\begin{aligned} n\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \\ \implies \int \cos^n x\,dx &= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx. \end{aligned}$$

(c)

$$\begin{aligned} \int \tan^n x\,dx &= \int \tan^2 x \tan^{n-2} x\,dx \\ &= \int (\sec^2 x - 1)\tan^{n-2} x\,dx \\ &= \int \sec^2 x \tan^{n-2} x\,dx - \int \tan^{n-2} x\,dx \\ &= \int \tan^{n-2} x\,d\tan x - \int \tan^{n-2} x\,dx \qquad [u(x) = \tan^x, du = \sec^2 x\,dx] \\ &= \int u^{n-2}\,du - \int \tan^{n-2} x\,dx \\ &= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\,dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx. \end{aligned}$$

(d)

$$\begin{aligned} \int \sec^n x\,dx &= \int \sec^2 x \sec^{n-2} x\,dx \qquad [u = \sec^{n-2} x, dv = \sec^2 x\,dx \implies v = \tan x] \\ &= \sec^{n-2} x \tan x - \int \tan x\,d\sec^{n-2} x \qquad [\int u\,dv = uv - \int v\,du] \\ &= \sec^{n-2} x \tan x - \int \tan x (n-2)\sec^{n-3} x \sec x \tan x\,dx \\ &= \sec^{n-2} x \tan x - (n-2)\int \tan^2 x \sec^{n-2} x\,dx \\ &= \sec^{n-2} x \tan x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x\,dx \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^n x - \sec^{n-2} x\,dx \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \end{aligned}$$

Then simplify and solve for $\int \sec^n x\,dx$, we have

$$\begin{aligned} (n-1)\int \sec^n x\,dx &= \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx \\ \implies \int \sec^n x\,dx &= \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx. \end{aligned}$$

$\square$

Theorem 1.3 (Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$).

(a) $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C.$

(b) $\displaystyle\int \cot x\,dx = \ln|\sin x| + C.$

(c) $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

(d) $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C.$

Proof.

(a)

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx && [u = \cos x, du = (-\sin x)\,dx]\\
&= -\int \frac{1}{\cos x}(-\sin x)\,dx\\
&= -\int \frac{1}{u}\,du\\
&= -\ln|u| + C\\
&= -\ln|\cos x| + C.
\end{aligned}$$

(b)

$$\begin{aligned}
\int \cot x\,dx &= \int \frac{\cos x}{\sin}\,dx && [u = \sin x, du = \cos x\,dx]\\
&= \int \frac{1}{\sin}\cos x\,dx\\
&= \int \frac{1}{u}\,du\\
&= \ln|u| + C\\
&= \ln|\sin x| + C.
\end{aligned}$$

(c) Observe that $\frac{d}{dx}\sec x = \sec x \tan x$ and $\frac{d}{dx}\tan x = \sec^2 x$. Then we have the following,

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx\\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx && [u = \sec x + \tan x, du = (\sec x \tan x + \sec^2 x)\,dx]\\
&= \int \frac{1}{\sec x + \tan x}(\sec^2 x + \sec x \tan x)\,dx\\
&= \int \frac{1}{u}\,du\\
&= \ln|u| + C\\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

(d) Note that $\dfrac{d}{dx}\csc x = -\cot x \csc x$ and $\dfrac{d}{dx}\cot x = -\csc^2 x$. Then we have

$$\begin{aligned}
\int \csc x\, dx &= \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}\, dx \\
&= \int \csc x \frac{(-1)(-\csc x - \cot x)}{\csc x + \cot x}\, dx \\
&= -\int \csc x \frac{(-\csc x - \cot x)}{\csc x + \cot x}\, dx \qquad [u = \csc x + \cot x, du = -\csc x \cot x - \csc^2 x\, dx] \\
&= -\int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x}\, dx \\
&= -\int \frac{1}{\csc x + \cot x} \cdot (-\csc^2 x - \csc x \cot x)\, dx \\
&= -\int \frac{1}{u}\, du \\
&= -\ln|u| + C \\
&= -\ln|\csc x + \cot x| + C.
\end{aligned}$$

$\square$

Procedure 1.3. Strategies for evaluating integrals of the form $\displaystyle\int \tan^m x \sec^n x\, dx$.

(a) When n is even. Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.
(b) When m is odd. Split off $\sec^2 x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.
(c) When m is even and n is odd. Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula to each term.

1.3 Trigonometric Substitutions

Procedure 1.4. Integrals Involving $a^2 - x^2$, $a^2 + x^2$ or $x^2 - a^2$

(a) The integral contains $a^2 - x^2$. Let $x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$ for $|x| \le a$. Then $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\cos^2\theta) = a^2\cos^2\theta$.
(b) The integral contains $a^2 + x^2$. Let $x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$.
(c) The integral contains $x^2 - a^2$. Let $x = a\sec\theta$, $\begin{cases} 0 \le \theta < \pi/2 & \text{for } x \ge a \\ \pi/2 < \theta \le \pi & \text{for } x \le -a \end{cases}$. Then $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$.

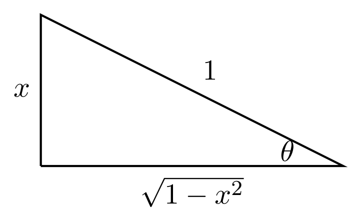
Example 1.1. Evaluate $\displaystyle\int_0^1 \sqrt{1-x^2}\, dx$.

Solution. Observe that both of U-Substitution and Integration by Parts are not applicable to this integral. Let's apply the trigonometric substitution to find the antiderivative and then apply Fundamental Theorem of Calculus to evaluate the definite integral. In this case, $a = 1$, then let

$x = \sin\theta, \theta \in [-\pi/2, \pi]$, and $dx = \cos\theta\, d\theta$. Then the indefinite integral becomes

$$\begin{aligned}
\int \sqrt{1-x^2}\, dx &= \int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta \\
&= \int \cos\theta\cos\theta\, d\theta \\
&= \int \cos^2\theta\, d\theta \\
&= \int \frac{1+2\cos 2\theta}{2}\, d\theta \\
&= \int \frac{1}{2}\, d\theta + \int \frac{\cos 2\theta}{2}\, d\theta \qquad\qquad [u = 2\theta, du = 2d\theta] \\
&= \frac{\theta}{2} + \frac{1}{4}\int \cos u\, du \\
&= \frac{\theta}{2} + \frac{1}{4}\sin u + C \\
&= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C. \\
&= \frac{\theta}{2} + \frac{1}{4}(2\sin\theta\cos\theta) + C. \\
&= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C.
\end{aligned}$$

Similar to U-Substitution, we need to substitute θ in terms of x. It is known that $x = \sin\theta \implies \theta = \arcsin x$ or $\theta = \sin^{-1} x$. And by $\sin^2 x + \cos^2 x = 1 \implies \cos x = \sqrt{1-\sin^2 x}$, we have $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. Or we can use geometry to find $\cos\theta$ as follows Then the antiderivative



becomes

$$\int \sqrt{1-x^2}\, dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

Now let us evaluate the definite integral using the above antiderivative:

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}\, dx &= \left.\frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}\right|_0^1 \\
&= \left(\frac{\arcsin 1}{2} + \frac{1\sqrt{1-1^2}}{2}\right) - \left(\frac{\arcsin 0}{2} + \frac{0\sqrt{1-0^2}}{2}\right) \\
&= \left(\frac{\pi/2}{2} + 0\right) - (0+0) \\
&= \frac{\pi}{4}.
\end{aligned}$$

$\square$