MATH 2205 - Calculus II Lecture Notes 06

Last update: June 20, 2019

1 Application of Integration

1.1 Reivew of General Slicing Method, Disk Method, and Washer Method

Definition 1.1 (General Slicing Method)**.** Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. Then volume of the solid is

$$V = \int_a^b A(x)\,dx.$$

Definition 1.2 (Disk Method about the x-Axis)**.** Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2\,dx.$$

Definition 1.3 (Washer Method about the x-Axis)**.** Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]\,dx.$$

Definition 1.4 (Disk and Washer Methods about the y-Axis)**.** Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi[p(y)^2 - q(y)^2]\,dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2\,dy.$$

Example 1.1 (Which solid has greater volume?)**.** Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

Solution. Solving $y^3 = 4y \implies y(y^2 - 4) = 0$, so the intersections in the first quadrant are $y = 0$, $y = 2$. Hence the corresponding points are $(0, 0)$ and $(8, 2)$. Therefore, the volume V_x of the region

R revolving about x-axis is

$$\begin{aligned} V_x &= \int_0^8 \pi \left[(x^{1/3})^2 - \left(\frac{x}{4}\right)^2 \right] dx \\ &= \pi \int_0^8 x^{2/3} - \frac{x^2}{16}\, dx \\ &= \pi \left(\frac{3x^{5/3}}{5} - \frac{x^3}{48} \right) \Big|_0^8 \\ &= \pi \left(\frac{38^{5/3}}{5} - \frac{8^3}{48} \right) \\ &= \pi \left(\frac{3 \times 32}{5} - \frac{32}{3} \right) \\ &= \frac{128\pi}{15}. \end{aligned}$$

On the other hand, the volume V_y of the region R revolving about y-axis is

$$\begin{aligned} V_y &= \int_0^2 \pi \left[(4y)^2 - (y^3)^2 \right] dy \\ &= \pi \int_0^2 16y^2 - y^6\, dy \\ &= \pi \left(\frac{16y^3}{3} - \frac{y^7}{7} \right) \Big|_0^2 \\ &= \pi \left(\frac{16 \times 2^3}{3} - \frac{2^7}{7} \right) \\ &= \pi \left(\frac{128}{3} - \frac{128}{7} \right) \\ &= \frac{512\pi}{21}. \end{aligned}$$

□

1.2 Volume by Shells

Definition 1.5. Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x [f(x) - g(x)]\, dx.$$

Example 1.2 (A sine bowl)**.** Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\pi/2}$. Find the volume of the solid generated when R is revolved about the y-axis.

Solution. The volume V of the region R about the y-axis:

$$
\begin{aligned}
V &= \int_0^{\sqrt{\pi/2}} 2\pi x[f(x) - 0]\,dx \\
&= \int_0^{\sqrt{\pi/2}} 2\pi x \sin x^2\,dx \\
&= \pi \int_0^{\sqrt{\pi/2}} \sin x^2 \cdot 2x\,dx \\
&= \pi \int_0^{\pi/2} \sin u\,du \\
&= \pi(-\cos u)\Big|_0^{\pi/2} \\
&= \pi[-0 - (-1)] \\
&= \pi.
\end{aligned}
$$

□

Example 1.3 (Shells about the x-axis)**.** Let R be the region in the first quadrant bounded by the graph of $y = \sqrt{x-2}$ and the line $y = 2$. Find the volume of the solid generated by when R is revolved about the x-axis.

$$
\begin{aligned}
V &= \int_0^2 2\pi y(y^2 + 2)\,dy \\
&= 2\pi \int_0^2 y^3 + 2y\,dy \\
&= 2\pi \left(\frac{y^4}{4} + y^2\right)\Bigg|_0^2 \\
&= 2\pi \left(\frac{2^4}{4} + 2^2\right) \\
&= 2\pi\,(4 + 4) \\
&= 16\pi.
\end{aligned}
$$

Example 1.4 (Volume of a drilled sphere)**.** A cylindrical hole with radius r is drilled symmetrically through the center of a sphere with radius a, where $0 \le r \le a$. What is the volume of the remaining material?

Solution. Noting that the equation for the sphere is $x^2 + y^2 = a^2 \implies y = \pm\sqrt{a^2 - x^2}$. The height of the rectangle is $h = \sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2}) = 2\sqrt{a^2 - x^2}$ while the width is the perimeter of the base which is a circle with radius x: $w = 2\pi x$. Then the volume from r to a can be obtained by applying shell method as follows

$$\begin{aligned}
V &= \int_r^a A(x)\,dx \\
&= \int_r^a 2\pi x(2\sqrt{a^2-x^2})\,dx \\
&= -2\pi \int_r^a \sqrt{\underbrace{a^2-x^2}_{u}}\,\underbrace{(-2x)\,dx}_{du} \\
&= -2\pi \int_{u(r)}^{u(a)} \sqrt{u}\,du \\
&= -2\pi \frac{2u^{3/2}}{3}\Bigg|_{u(r)=a^2-r^2}^{u(a)=a^2-a^2=0} \\
&= -2\pi \left[0 - \frac{2(a^2-r^2)^{3/2}}{3}\right] \\
&= \frac{4\pi(a^2-r^2)^{3/2}}{3}.
\end{aligned}$$

□

Example 1.5 (Revolving about other lines)**.** Let R be the region bounded by the curve $y=\sqrt{x}$, the line $y=1$, and the y-axis.

(a) Use the shell method to find the volume of the solid generated when R is revolved about the line $x=-\frac{1}{2}$.
(b) Use the disk/washer method to find the volume of the solid generated when R is revolved about the line $y=1$.

Solution.

(a) The height of the rectangle is $h=1-\sqrt{x}$, and the width of the rectangle is the circumference of the base which is a circle centered at $x=-\frac{1}{2}$ and thus with radius $x-\left(-\frac{1}{2}\right)=x+\frac{1}{2}$. Then we have the volume of the region revolved about the line $x=-\frac{1}{2}$ as follows,

$$\begin{aligned}
V &= \int_0^1 2\pi\left(x+\frac{1}{2}\right)(1-\sqrt{x})\,dx \\
&= 2\pi\int_0^1 x - x^{3/2} + \frac{1}{2} - \frac{x^{1/2}}{2}\,dx \\
&= 2\pi\left(\frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x}{2} - \frac{x^{3/2}}{3}\right)\Bigg|_0^1 \\
&= 2\pi\left(\frac{1}{2} - \frac{2}{5} + \frac{1}{2} - \frac{1}{3}\right) \\
&= \frac{8\pi}{15}.
\end{aligned}$$

(b) Slice the solid at the plane $x=x$, the corresponding cross section is a circle centered at $y=1$ with radius $1-\sqrt{x}$, then the volume of the solid generated when R is revolved about the line

$y = 1$ is

$$\begin{aligned} V &= \int_0^1 \pi(1-\sqrt{x})^2\,dx \\ &= \pi\int_0^1 1 - 2\sqrt{x} + x\,dx \\ &= \pi\left(x - \frac{4x^{3/2}}{3} + \frac{x^2}{2}\right)\Bigg|_0^1 \\ &= \pi\left(1 - \frac{4}{3} + \frac{1}{2}\right) \\ &= \frac{\pi}{6}. \end{aligned}$$

□

1.3 Length of Curves

Definition 1.6 (Arc Length for $y = f(x)$)**.** Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx.$$

Example 1.6 (Arc length)**.** Find the length of the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 4$.

Solution. By the definition, we have the arc length L below

$$\begin{aligned} L &= \int_0^4 \sqrt{1 + f'(x)^2}\,dx \\ &= \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^4 \Bigg(\underbrace{1 + \frac{9}{4}x}_{u}\Bigg)^{1/2} \frac{4}{9}\cdot\underbrace{\frac{9}{4}\,dx}_{du} \\ &= \frac{4}{9}\int_{u(0)}^{u(4)} u^{1/2}\,du \\ &= \frac{4}{9}\frac{3u^{3/2}}{2}\Bigg|_{u(0)=1}^{u(4)=10} \\ &= \frac{2}{3}\left(10^{3/2} - 1\right) \end{aligned}$$

□

Definition 1.7 (Arc Length for $x = g(y)$)**.** Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. Then length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\,dy.$$

1.4 Surface Area

Definition 1.8 (Area of a Surface of Revolution)**.** Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$