MATH 2205 - Calculus II Lecture Notes 04

Last update: June 20, 2019

1 Integration

1.1 Reivew of Mean Value Theorem and Substitution Rule for Integrals

Theorem 1.1 (Mean Value Theorem for Integrals)**.** Let f continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Theorem 1.2 (Substitution Rule for Indefinite Integrals)**.** Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Theorem 1.3 (Substitution Rule for Definite Integrals)**.** Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

1.2 Substitution Rule using Properties of Trig Functions

Proposition 1.1 (Properties of Trig Functions)**.**

(a) $\sin^2\theta + \cos^2\theta = 1$.
(b) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
(c) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
(d) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.
(e) $\sin 2\theta = 2\sin\theta\cos\theta$.
(f) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.
(g) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.
(h) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

Example 1.1 (Trig Functions)**.** Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/2} \cos^2\theta\,d\theta$.
(b) $\displaystyle\int_0^{\pi/2} \cos^3 t\,dt$.
(c) $\displaystyle\int_0^{\pi/2} \cos^4\alpha\,d\alpha$.

Solution.

(a)

$$\begin{aligned}
\int_0^{\pi/2} \cos^2\theta\, d\theta &= \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\, d\theta + \int_0^{\pi/2} \underbrace{\frac{\cos 2\theta}{4}}_{\cos u/4} \cdot \underbrace{2\, d\theta}_{du} \\
&= \frac{1}{2}\theta\Big|_0^{\pi/2} + \frac{1}{4}\int_{u(0)}^{u(\pi/2)} \cos u\, du \\
&= \frac{1}{2}\frac{\pi}{2} + \frac{1}{4}\int_0^{\pi} \cos u\, du && [u(0) = 2\times 0 = 0, u(\pi/2) = 2\times\frac{\pi}{2} = \pi] \\
&= \frac{\pi}{4} + \frac{1}{4}\sin u\Big|_0^{\pi} && [\text{Antiderivative}] \\
&= \frac{\pi}{4} + \frac{1}{4}(\sin\pi - \sin 0) \\
&= \frac{\pi}{4}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\pi/2} \cos^3 t\, dt &= \int_0^{\pi/2} \cos^2 t \cos t\, dt \\
&= \int_0^{\pi/2} \underbrace{(1-\sin^2 t)}_{1-u^2} \underbrace{\cos t\, dt}_{du} \\
&= \int_{u(0)}^{u(\pi/2)} (1-u^2)\, du && [\text{Substitute } u = \sin t,\ du = \cos t\, dt] \\
&= \int_0^1 (1-u^2)\, du && [u(0) = \sin 0 = 0, u(\pi/2) = \sin\pi/2 = 1] \\
&= \int_0^1 1\, du - \int_0^1 u^2\, du \\
&= u\Big|_0^1 - \frac{1}{3}u^3\Big|_0^1 && [\text{Antiderivative}] \\
&= 1 - \frac{1}{3}(1^3 - 0^3) \\
&= \frac{2}{3}.
\end{aligned}$$

(c)

$$
\begin{aligned}
\int_0^{\pi/2} \cos^4 \alpha \, d\alpha &= \int_0^{\pi/2} \cos^2 \alpha \cdot \cos^2 \alpha \, d\alpha \\
&= \int_0^{\pi/2} \frac{1+\cos 2\alpha}{2} \frac{1+\cos 2\alpha}{2} \, d\alpha \\
&= \int_0^{\pi/2} \frac{1 + 2\cos 2\alpha + \cos^2 2\alpha}{4} \, d\alpha \\
&= \frac{1}{4}\int_0^{\pi/2} d\alpha + \frac{1}{2}\int_0^{\pi/2} \cos 2\alpha \, d\alpha + \frac{1}{4}\int_0^{\pi/2} \cos^2 2\alpha \, d\alpha \\
&= \frac{1}{4}\alpha \Big|_0^{\pi/2} + \frac{1}{4}\int_0^{\pi/2} \underbrace{\cos 2\alpha}_{\cos u} \cdot \underbrace{2\,d\alpha}_{du} + \frac{1}{4}\int_0^{\pi/2} \frac{1+\cos 4\alpha}{2} \, d\alpha \\
&= \frac{\pi}{8} + \frac{1}{4}\int_{u(0)}^{u(\pi/2)} \cos u \, du + \frac{1}{8}\int_0^{\pi/2} d\alpha + \frac{1}{8}\frac{1}{4}\int_0^{\pi/2} \underbrace{\cos 4\alpha}_{\cos v} \cdot \underbrace{4\,d\alpha}_{dv} \\
&= \frac{\pi}{8} + \frac{1}{4}\int_0^{\pi} \cos u \, du + \frac{1}{8}\alpha \Big|_0^{\pi/2} + \frac{1}{32}\int_{v(0)}^{v(\pi/2)} \cos v \, dv \\
&= \frac{\pi}{8} + \frac{1}{4}\sin u \Big|_0^{\pi} + \frac{\pi}{16} + \frac{1}{32}\sin v \Big|_{v(0)=0}^{v(\pi/2)=2\pi} \\
&= \frac{\pi}{8} + 0 + \frac{\pi}{16} + 0 \\
&= \frac{3\pi}{16}.
\end{aligned}
$$

□

2 Application of Integration

2.1 Regions Between Curves

Definition 2.1 (Approximation of Area of a Region Between Curves using Riemman Sum)**.** Suppose that f and g continuous on an interval $[a, b]$ on which $f(x) \geq g(x)$. Partition the interval $[a, b]$ into n subintervals using uniformly spaced grid points separated by a distance $\Delta x = (b - a)/n$. Then the area A of the region bounded by the two curves and the vertical lines $x = a$ and $x = b$ can be approximated by:

$$A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x,$$

where $x_k^* \in [x_{k-1}, x_k], k = 1, 2, \ldots, n, x_0 = a, x_n = b$.

Definition 2.2 (Area of a Region Between Two Curves)**.** Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)] \, dx = \lim_{n\to\infty} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x.$$

Example 2.1 (Area between curves)**.** Find the area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$ on the interval on which $f(x) \geq g(x)$.

Solution. First we need to find the interval on which $f(x) \geq g(x)$ by solving $f(x) = g(x)$, i.e.,

$$5 - x^2 = x^2 - 3 \implies 2x^2 = 8 \implies x^2 = 4 \implies x = \pm 2.$$

With the help of the graphs of $f(x)$ and $g(x)$, we can determine the interval is $[-2, 2]$. Then the area A of the region is

$$\begin{aligned}
A &= \int_{-2}^{2} f(x) - g(x)\,dx \\
&= \int_{-2}^{2} [(5 - x^2) - (x^2 - 3)]\,dx \\
&= \int_{-2}^{2} -2x^2 + 8\,dx \\
&= 2\int_{0}^{2} -2x^2 + 8\,dx && [-2x^2 + 8 \text{ is even}] \\
&= 2\left(\frac{-2x^3}{3} + 8x\right)\Bigg|_0^2 && [\text{FTOC}] \\
&= 2\left(\frac{-2 \times 2^3}{3} + 8 \times 2\right) \\
&= 2 \times \frac{32}{3} \\
&= \frac{64}{3}.
\end{aligned}$$

□

Example 2.2 (Compound region)**.** Find the area of the region bounded by the graphs of $f(x) = -x^2 + 3x + 6$ and $g(x) = |2x|$ on the interval on which $f(x) \geq g(x)$.

Solution. The lower boundary of the region is bounded by two different branches of the absolute value function. Then we need to divide the region into two (or more) subregions whose areas are found independently and then summed. By the definition of absolute value,

$$g(x) = |2x| = \begin{cases} g_-(x) = -2x & \text{if } x < 0, \\ g_+(x) = 2x & \text{if } x \geq 0. \end{cases}$$

Next let us find the corresponding interval by solving $f(x) = g(x)$,

$$\begin{aligned}
&\begin{cases} -x^2 + 3x + 6 = -2x & \text{if } x < 0, \\ -x^2 + 3x + 6 = 2x & \text{if } x \geq 0, \end{cases} \implies \begin{cases} -x^2 + 5x + 6 = (-x + 6)(x + 1) = 0 & \text{if } x < 0, \\ -x^2 + 1x + 6 = (-x + 3)(x + 2) = 0 & \text{if } x \geq 0, \end{cases} \\
&\implies \begin{cases} x = 6 \text{ (does not satisfy } x < 0\text{, ruled out)}, x = -1, \\ x = 3, x = -2 \text{ (does not satisfy } x \geq 0\text{, ruled out)}, \end{cases} \\
&\implies [a, b] = [-1, 3].
\end{aligned}$$

Then the intervals for two subregions are $[-1, 0]$ and $[0, 3]$. Then the area A of the region is

$$\begin{aligned}
A &= \int_{-1}^{0} [f(x) - g_-(x)]\, dx + \int_{0}^{3} [f(x) + g_+(x)]\, dx \\
&= \int_{-1}^{0} [(-x^2 + 3x + 6) - (-2x)]\, dx + \int_{0}^{3} [(-x^2 + 3x + 6) - (2x)]\, dx \\
&= \int_{-1}^{0} -x^2 + 5x + 6\, dx + \int_{0}^{3} -x^2 + x + 6\, dx \\
&= \left(-\frac{x^3}{3} + \frac{5x^2}{2} + 6x\right)\Bigg|_{-1}^{0} + \left(-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right)\Bigg|_{0}^{3} \\
&= \left(-\frac{1}{3} - \frac{5}{2} + 6\right) + \left(-\frac{3^3}{3} + \frac{3^2}{2} + 18\right) \\
&= \frac{19}{6} + \frac{27}{2} \\
&= \frac{50}{3}.
\end{aligned}$$

$\square$