MATH 2205 - Calculus II Lecture Notes 03

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1 Integration

1.1 Reivew of FTOC, Integrals of Even and Odd Functions

Definition 1.1 (Area Function)**.** Let f be a continuous function, for $t \geq a$. The *area function* for f with left endpoint a is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Theorem 1.1 (Fundamental Theorem of Calculus (FTOC), Part I)**.** If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b.$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Theorem 1.2 (Fundamental Theorem of Calculus (FTOC), Part II)**.** If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

Theorem 1.3 (Integrals of Even and Odd Functions)**.** Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

$$\int_{-a}^a f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even,} \\ 0 & \text{if } f \text{ is odd.} \end{cases}$$

1.2 Working with Integrals

Definition 1.2 (Average Value of a Function)**.** The average value of an integrable function f on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem 1.4 (Mean Value Theorem for Integrals)**.** Let f continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Example 1.1 (Average value equals function value)**.**

(a) Find the point(s) on the interval $(0, 1)$ at which $f(x) = 2x(1-x)$ equals its average value on $[0, 1]$.
(b) Find the point(s) on the interval $(0, 2)$ at which $g(x) = x^2 + 1$ equals its average value on $[0, 2]$.

Solution. (a) The average value of f on $[0, 1]$ is

$$\begin{aligned}
\overline{f} &= \frac{1}{1-0}\int_0^1 2x(1-x)\,dx \\
&= \int_0^1 2x - 2x^2\,dx \\
&= 2\int_0^1 x\,dx - 2\int_0^1 x^2\,dx && [\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx] \\
&= 2\frac{x^2}{2}\bigg|_0^1 - 2\frac{x^3}{3}\bigg|_0^1 && [\text{Antiderivative}] \\
&= 2\left(\frac{1}{2}-0\right) - 2\left(\frac{1}{3}-0\right) \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3}.
\end{aligned}$$

Next let us find the points on $[0, 1]$ at which $f(x) = 2x(1-x) = 1/3 \implies -2x^2 + 2x - \frac{1}{3} = 0$. Using the quadratic formula, i.e., $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ is the root of $ax^2 + bx + c = 0$, we have

$$x = \frac{-2 \pm \sqrt{4 - 8/3}}{-4} = \frac{-2 \pm \sqrt{4/3}}{-4} = \frac{1 \pm \sqrt{1/3}}{2}.$$

Therefore, we have $x_1 = \frac{1+\sqrt{1/3}}{2}$ and $x_2 = \frac{1-\sqrt{1/3}}{2}$. Note that $0 < x_1, x_2 < 1$, we have two points, namely $x_1 = \frac{1+\sqrt{1/3}}{2}$, $x_2 = \frac{1-\sqrt{1/3}}{2}$ on the interval $(0, 1)$ at which $f(x) = 2x(1-x)$ equals its average value on $[0, 1]$.

(b) The average value of g on $[0, 2]$ is

$$\begin{aligned}\overline{g} &= \frac{1}{2-0}\int_0^2 x^2 + 1\,dx \\ &= \frac{1}{2}\left(\frac{x^3}{3} + x\right)\Bigg|_0^2 \\ &= \frac{1}{2}\left(\frac{2^3}{3} + 2 - 0\right) \\ &= \frac{7}{3}.\end{aligned}$$

Next find the points x such that $g(x) = x^2 + 1 = 7/3$. Solve for x, we have

$$x^2 = \frac{4}{3} \implies x = \pm\sqrt{\frac{4}{3}} = \pm\sqrt{\frac{2^2}{3}} = \pm 2\sqrt{\frac{1}{3}}.$$

Then we have $x_1 = 2\sqrt{\frac{1}{3}}$ and $x_2 = -2\sqrt{\frac{1}{3}}$. Since $0 < x_1 = 2\sqrt{\frac{1}{3}} < 2$, $x_2 = -2\sqrt{\frac{1}{3}} < 0$, x_1 is the only desired point on the interval $(0, 2)$ at which $g(x_1) = x_1^2 + 1$ equals its average value on $[0, 2]$.

□

1.3 Substitution Rule

Theorem 1.5 (Substitution Rule for Indefinite Integrals)**.** Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Procedure 1.1 (Substitution Rule (Change of Variables))**.**

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example 1.2 (Perfect substitution)**.** Use the Substitution Rule to find the following indefinite integrals. Check your work by differentiating.

(a) $\displaystyle\int 2(2x+1)^3\,dx.$

(b) $\displaystyle\int 10e^{10x}\,dx.$

Solution.

(a) We identify $u = 2x + 1$ as the inner function of the composite function $(2x + 1)^3$. Therefore, we choose the new variable $u = 2x + 1$, which implies that $\frac{du}{dx} = 2$, or $du = 2dx$. Notice that

$du = 2dx$ appears as a factor in the integrand. The change of variables looks like this:

$$\begin{aligned}
\int 2(2x+1)^3\,dx &= \int \underbrace{(2x+1)^3}_{u^3} \cdot \underbrace{2\,dx}_{du} \\
&= \int u^3 du && \text{[Substitute } u = 2x+1, du = 2\,dx\text{]} \\
&= \frac{u^4}{4} + C && \text{[Antiderivative]} \\
&= \frac{(2x+1)^4}{4} + C. && \text{[Replace } u \text{ with } 2x+1\text{]}
\end{aligned}$$

(b) The composite function e^{10x} has the inner function $u = 10x$, which implies that $du = 10dx$. The change of variable appears as

$$\begin{aligned}
\int 10e^{10x}\,dx &= \int \underbrace{e^{10x}}_{e^u} \cdot \underbrace{10\,dx}_{du} \\
&= \int e^u\,du && \text{[Substitute } u = 10x,\ du = 10\,dx\text{]} \\
&= e^u + C && \text{[Antiderivative]} \\
&= e^{10x} + C. && \text{[Replace } u \text{ with } 10x\text{]}
\end{aligned}$$

$\square$

Example 1.3 (Introducing a constant)**.** Find the following indefinite integrals.

(a) $\displaystyle\int x^4(x^5+6)^9\,dx.$

(b) $\displaystyle\int \cos^3 x \sin x\,dx.$

Solution.

(a)

$$\begin{aligned}
\int x^4(x^5+6)^9\,dx &= \int \underbrace{(x^5+6)^9}_{u^9} \cdot \frac{1}{5} \underbrace{5x^4\,dx}_{du} \\
&= \int u^9 \cdot \frac{1}{5}\,du && \text{[Substitute } u = x^5+6,\ du = 5x^4\,dx\text{]} \\
&= \frac{1}{5}\int u^9\,du && [\textstyle\int cf(x)\,dx = c\int f(x)\,dx] \\
&= \frac{1}{5}\frac{u^{10}}{10} + C && \text{[Antiderivative]} \\
&= \frac{1}{50}u^{10} + C \\
&= \frac{1}{50}(x^5+6)^{10} + C. && \text{[Replace } u \text{ with } x^5+6\text{]}
\end{aligned}$$

(b)

$$
\begin{aligned}
\int \cos^3 x \sin x\, dx &= \int \underbrace{\cos^3 x}_{u^3} \cdot (-1) \underbrace{-\sin x\, dx}_{du} \\
&= \int u^3 \cdot (-1)\, du && [\text{Substitute } u = \cos x,\ du = -\sin x\, dx] \\
&= -\int u^3\, du && [\int cf(x)\, dx = c \int f(x)\, dx] \\
&= -\frac{u^4}{4} + C && [\text{Antiderivative}] \\
&= -\frac{\cos^4 x}{4} + C. && [\text{Replace } u \text{ with } \cos x]
\end{aligned}
$$

□

Example 1.4 (Variation on the substitution method)**.** Find $\displaystyle\int \frac{x}{\sqrt{x+1}}\, dx$.

SOLUTION.

$$
\begin{aligned}
\int \frac{x}{\sqrt{x+1}}\, dx &= \int \frac{u-1}{\sqrt{u}}\, du && [\text{Substitute } u = x+1,\ du = dx] \\
&= \int (u-1)u^{-1/2}\, du && [\frac{1}{\sqrt{u}} = u^{-1/2}] \\
&= \int u^{1/2} - u^{-1/2}\, du \\
&= \int u^{1/2}\, du - \int u^{-1/2}\, du && [\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx] \\
&= \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C && [\text{Antiderivative}] \\
&= \frac{2}{3}u^{3/2} - 2u^{1/2} + C \\
&= \frac{2}{3}u^{1/2} - 2u^{1/2} + C \\
&= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C. && [\text{Replace } u \text{ with } x+1]
\end{aligned}
$$

□

Theorem 1.6 (Substitution Rule for Definite Integrals)**.** Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example 1.5 (Definite integrals)**.** Evaluate the following integrals.

(a) $\displaystyle\int_0^2 \frac{dx}{(x+3)^3}$.

(b) $\displaystyle\int_0^4 \frac{x}{x^2+1}\, dx$.

(c) $\displaystyle\int_0^{\pi/2} \sin^4 x \cos x \, dx.$

Solution.

(a)

$$\begin{aligned}
\int_0^2 \frac{dx}{(x+3)^3} &= \int_{u(0)}^{u(2)} \frac{1}{u^3}\, du && [\text{Substitute } u = x+3,\ du = dx] \\
&= \int_3^5 u^{-3}\, du && [u(0) = 0+3, u(2) = 2+3, \frac{1}{u^3} = u^{-3}] \\
&= \left.\frac{u^{-2}}{-2}\right|_3^5 \\
&= -\frac{1}{2}u^{-2}\Big|_3^5 \\
&= -\frac{1}{2}(5^{-2} - 3^{-2}) \\
&= -\frac{1}{2}\left(\frac{1}{25} - \frac{1}{9}\right) \\
&= \frac{8}{225}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^4 \frac{x}{x^2+1}\, dx &= \int_0^4 \underbrace{\frac{1}{x^2+1}}_{1/u} \cdot \frac{1}{2} \underbrace{2x\, dx}_{du} \\
&= \frac{1}{2}\int_{u(0)}^{u(4)} \frac{1}{u}\, du && [\text{Substitute } u = x^2+1,\ du = 2x\, dx] \\
&= \frac{1}{2}\int_1^{17} u^{-1}\, du && [u(0) = 0^2+1 = 1, u(4) = 4^2+1 = 17, \frac{1}{u} = u^{-1}] \\
&= \frac{1}{2}\ln|u|\Big|_1^{17} && [\text{Antiderivative}] \\
&= \frac{1}{2}(\ln 17 - \ln 1) \\
&= \frac{\ln 17}{2}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^{\pi/2} \underbrace{\sin^4 x}_{u^4} \underbrace{\cos x\,dx}_{du} &= \int_{u(0)}^{u(\pi/2)} u^4\,du && [\text{Substitute } u = \sin x,\ du = \cos x\,dx] \\
&= \int_0^1 u^4\,du && [u(0) = \sin 0 = 0, u(\pi/2) = \sin \pi/2 = 1] \\
&= \frac{1}{5}u^5 \Big|_0^1 && [\text{Antiderivative}] \\
&= \frac{1}{5}(1^5 - 0^5) \\
&= \frac{1}{5}.
\end{aligned}$$

$\square$