MATH 3341: Introduction to Scientific Computing Lab

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Lab 11: MATLAB Integration Routines & Gauss Quadrature

Built-in Integration Routines

`polyint` Integrate polynomial analytically

- `polyint(P,K)` returns a polynomial representing the integral of polynomial P, using a scalar constant of integration K.
- `polyint(P)` assumes a constant of integration K=0.

`trapz` Trapezoidal numerical integration

`Z = trapz(X,Y)` computes the integral of Y with respect to X using the trapezoidal method. X and Y must be vectors of the same length, or X must be a column vector and Y an array whose first non-singleton dimension is length(X). trapz operates along this dimension.

Let $X = [x_1, x_2, \ldots, x_n]$, $Y = [y_1, y_2, \ldots, y_n]$, then

$$Z = \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)(y_{i+1} + y_i)}{2} = \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)(y_{i+1} + y_i).$$

`trapz` Trapezoidal numerical integration

- `Z = trapz(Y)` computes an approximation of the integral of Y via the trapezoidal method (with unit spacing). To compute the integral for spacing different from one, multiply Z by the spacing increment.
- `Z = trapz(X,Y,DIM)` or `trapz(Y,DIM)` integrates across dimension DIM of Y. The length of X must be the same as size(Y,DIM)).

`cumtrapz` Cumulative trapezoidal numerical integration

- `Z = cumtrapz(Y)` computes an approximation of the cumulative integral of Y via the trapezoidal method (with unit spacing). To compute the integral for spacing different from one, multiply Z by the spacing increment.
- `Z = cumtrapz(X,Y)` computes the cumulative integral of Y with respect to X using trapezoidal integration. X and Y must be vectors of the same length, or X must be a column vector and Y an array whose first non-singleton dimension is length(X). cumtrapz operates across this dimension.
- `Z = cumtrapz(X,Y,DIM)` or `cumtrapz(Y,DIM)` integrates along dimension DIM of Y. The length of X must be the same as size(Y,DIM)).

`integral` Numerically evaluate integral.

- `Q = integral(FUN,A,B)` approximates the integral of function FUN from A to B using global adaptive quadrature and default error tolerances.
  FUN must be a function handle. A and B can be -Inf or Inf. If both are finite, they can be complex. If at least one is complex, integral approximates the path integral from A to B over a straight line path.
- `Q = integral(FUN,A,B,PARAM1,VAL1,PARAM2,VAL2,...)` performs the integration with specified values of optional parameters.

`integral2` Numerically evaluate double integral

- `Q = integral2(FUN,XMIN,XMAX,YMIN,YMAX)` approximates the integral of FUN(X,Y) over the planar region XMIN $<=$ X $<=$ XMAX and YMIN(X) $<=$ Y $<=$ YMAX(X). FUN is a function handle, YMIN and YMAX may each be a scalar value or a function handle.
- `Q = integral2(FUN,XMIN,XMAX,YMIN,YMAX,PARAM1,VAL1,PARAM2,VA` performs the integration as above with specified values of optional parameters

`integral3` Numerically evaluate triple integral

- `Q = integral3(FUN,XMIN,XMAX,YMIN,YMAX,ZMIN,ZMAX)` approximates the integral of FUN(X,Y,Z) over the region XMIN <= X <= XMAX, YMIN(X) <= Y <= YMAX(X), and ZMIN(X,Y) <= Z <= ZMAX(X,Y). FUN is a function handle, YMIN, YMAX, ZMIN, and ZMAX may each be a scalar value or a function handle.
- `Q = integral3(FUN,XMIN,XMAX,YMIN,YMAX,ZMIN,ZMAX,PARAM1,VAL1, PARAM2,VAL2,...)` performs the integration as above with specified values of optional parameters

Gauss-Legendre Quadrature

Gauss-Legendre Quadrature on $[-1, 1]$

Integration of $f(x)$ on the interval $[-1, 1]$ using Gauss Quadrature is given by

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where w_i and x_i are chosen so the integration rule is exact for the largest class of polynomials. $f(x)$ is well-approximated by polynomial on $[-1, 1]$, the associated orthogonal polynomials are *Legendre polynomial*, denoted by $P_n(x)$. With the n-th polynomial normalized to give $P_n(1) = 1$, the i-th Gauss node, x_i, is the i-th root of P_n and the weights are given by the formula (Abramowitz & Stegun 1972, p. 887):

$$w_i = \frac{2}{(1 - x_i^2)[P_n'(x_i)]^2}.$$

Gauss-Legendre Quadrature on $[a, b]$

To approximate the integral on the general interval $[a, b]$, we need to use the change of variables as follows:

$$\frac{t-a}{b-a} = \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2} \implies t = \frac{b-a}{2}x + \frac{b+a}{2}, -1 \le x \le 1$$

$$\implies dt = \frac{b-a}{2}dx.$$

So the Gauss Quadrature on a general interval $[a, b]$ is given by

$$\int_a^b f(t)\,dt = \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}\,dx$$

$$\approx \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)\frac{b-a}{2}.$$

Gauss-Legendre Quadrature on $[a, b]$

Let

$$g(x) = f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2},$$

then

$$\int_a^b f(t)\,dt = \int_{-1}^1 g(x)\,dx \approx \sum_{i=1}^n w_i g(x_i).$$