MATH 5290 - Operator Algebras & K-Theory Lecture Notes 1

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1 Hilbert Space

1.1 Hilbert Space

Definition 1.1 (Pre-inner product space)**.** A pre-inner product space is a vector space together with an pre-inner product $\langle\cdot,\cdot\rangle : \mathcal{H} \to \mathbb{C}$ such that

(1) $\langle \alpha\xi + \beta\eta, \zeta\rangle = \alpha\langle\xi,\zeta\rangle + \beta\langle\eta,\zeta\rangle, \forall \alpha, \beta \in \mathbb{C}, \xi, \eta, \zeta \in \mathcal{H}$.

(2) $\langle\xi,\eta\rangle = \overline{\langle\eta,\xi\rangle}$.

(3) $\langle\xi,\xi\rangle \geq 0$.

Definition 1.2 (Inner product space)**.** A pre-inner product space is called an inner product space if $\langle\xi,\xi\rangle = 0 \iff \xi = 0$.

Lemma 1.1. Define $\|\xi\| = \sqrt{\langle\xi,\xi\rangle}, \forall \xi \in \mathcal{H}$, then $\|\cdot\|$ is a pre-norm. If $\langle\cdot,\cdot\rangle$ is a inner product, then $\|\cdot\|$ is a norm.

Definition 1.3 (Hilbert space)**.** A Hilbert space is a inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ such that $\mathcal{H}$ is complete under $\|\cdot\|$.

Question: How do we construct an inner product space from a pre-inner product space? Consider $\mathcal{H}/N$, where $N = \{\xi : \langle\xi,\xi\rangle = 0\}$. However, N needs to be a subspace. Here is why it is: $\forall \xi, \eta \in \mathcal{H}$, we have

$$\langle\xi+\eta,\xi+\eta\rangle = \langle\xi,\xi\rangle + \langle\xi,\eta\rangle + \langle\eta,\xi\rangle + \langle\eta,\eta\rangle = 2\operatorname{Re}\langle\xi,\eta\rangle.$$

By Cauchy-Schwarz inequality, $|\langle\xi,\eta\rangle|^2 \leq \langle\xi,\xi\rangle\langle\eta,\eta\rangle = 0$. Pick $\lambda \in \mathcal{T}$ such that $\langle\lambda\xi,\eta\rangle \in \mathbb{R}$, that is, $\langle\lambda\xi,\eta\rangle = \operatorname{Re}\langle\lambda\xi,\eta\rangle$. Then, we can obtain the following

$$\begin{aligned}
|\operatorname{Re}\langle\lambda\xi,\eta\rangle|^2 &\leq \langle\lambda\xi,\lambda\xi\rangle\langle\eta,\eta\rangle \\
|\lambda\langle\xi,\eta\rangle|^2 &\leq |\lambda|^2\langle\xi,\xi\rangle\langle\eta,\eta\rangle \\
|\lambda|^2|\langle\xi,\eta\rangle|^2 &\leq |\lambda|^2\langle\xi,\xi\rangle\langle\eta,\eta\rangle \\
|\langle\xi,\eta\rangle|^2 &\leq \langle\xi,\xi\rangle\langle\eta,\eta\rangle.
\end{aligned}$$

Example 1.1.

(1) $\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n), z_i \in \mathbb{C}\} = \{f : \{1, \ldots, n\} \to \mathbb{C}\}$ with $\langle(z_1, z_2, \ldots, z_n), (w_1, w_2, \ldots, w_n)\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$. Note: $X^Y = \{f : Y \to X\}$.

(2) Let X be a set. $\ell^2(X) = \{f : X \to \mathbb{C} : \sum_{x\in X} |f(x)|^2 < \infty\}$. Note: $\sum_{x\in X} |f(x)|^2 < \infty \iff \exists M \in \mathbb{R}, \forall$ finite set $S \subset X$, $\sum_{x\in S} |f(x)|^2 < M$. Define the inner product as follows, $\langle f, g\rangle = \sum_{x\in X} f(x)\cdot\overline{g}(x)$.

(3) Let $X = G$, where G is a discrete group. $\ell^2(G)$.

Remark 1.1. Any Hilbert space is isomorphic to $\ell^2(X)$ for some X.

1.2 Orthogonality

Definition 1.4 (Orthogonality)**.** $\xi \perp \eta \iff \langle \xi, \eta \rangle = 0$.

Lemma 1.2. Let S be a subset of $\mathcal{H}$, $S^\perp = \{\xi \in \mathcal{H} : \xi \perp \eta, \eta \in S\}$. Then $S^\perp$ is a subspace, where $S^\perp = (\operatorname{span} S)^\perp$.

Proposition 1.1. Assume that S is a closed subspace, then $\mathcal{H} \cong S \oplus S^\perp$.

Proof. i.e., $\forall \xi \in \mathcal{H}, \exists! \xi_0, \xi_1$ such that $\xi_0 \in S$ and $\xi_1 \in S^\perp$ such that $\xi = \xi_0 + \xi_1$. Uniqueness: $\xi_0 + \xi_1 = \xi_0' + \xi_1' \implies \xi_0 - \xi_0' = \xi_1' - \xi_1$, where $\xi_0 - \xi_0' \in S$ and $\xi_1' - \xi_1 \in S^\perp$. That implies $\xi_0 - \xi_0' = \xi_1' - \xi_1 = 0$.
Fact:

(1) $\exists! \xi_0$ such that $\|\xi - \xi_0\| = \inf_{\eta \in S}\{\|\xi - \eta\|\}$.

(2) $(\xi - \xi_0) \perp S$.

□

Definition 1.5. The mapping $\xi \mapsto \xi_0$ is called the orthogonal projection onto S.

Definition 1.6. $\{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \subset \mathcal{H}$ is an orthonormal family if

(1) $\|\xi_i\| = 1, i = 1, 2, \ldots, n$.

(2) $\xi_i \perp \xi_j$, where $i \neq j$.

Remark 1.2. An orthonormal basis is a maximal orthonormal family.

Example 1.2. $\ell^2(\mathbb{Z})$, the basis are $\{e_i = (0, 0, \ldots, 1, \ldots 0)\}$.

Definition 1.7. Suppose that $\{e_n\} \subset \mathcal{H}$ is an orthonormal set of $\mathcal{H}$ if

(1) $\|e_n\| = 1$.

(2) $e_i \perp e_j, i \neq j$.

Then $\{e_n\}$ is an orthonormal basis if it is maximal.

Theorem 1.1. Let $\{e_n, n \in \Lambda\}$ be an orthonormal basis. $\forall \xi \in \mathcal{H}$, one has

$$\xi = \sum_{n \in \Lambda} \langle \xi, e_n \rangle e_n.$$

Then

$$\|\xi\| = \sqrt{\sum_{n \in \Lambda} |\langle \xi, e_n \rangle^2}.$$

Corollary 1.1.

$$\mathcal{H} \cong \ell^2(S).$$

1.3 Topologies on a Hilbert Space

Definition 1.8 (Norm topology)**.** Let $\|\xi\|^2 = \langle \xi, \xi \rangle$, then $\xi_n \to \xi \iff \|\xi_n - \xi\| \to 0$.

In $\mathbb{C}^n$, the unit ball $\{\xi : \|\xi\| \leq 1\}$ is compact. In particular, any sequence in the unit ball has a convergent subsequence. On the other hand, in the infinite dimensional space, the unit ball $\{\xi : \|\xi\| \leq 1\}$ is closed and bounded, however, NOT compact!
Consider $\mathcal{H} = \ell^2(\mathbb{N})$, pick the orthonormal basis $e_n = (0, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$. Then, $\{e_n\}_{n=1}^\infty \subset B_1$, for any $m \neq n$, we have $\|e_m - e_n\| = \sqrt{2}$, therefore, no subsequence would converge.

Definition 1.9 (Weak topology)**.** Dual space of a normed space. Let $(X, \|\cdot\|)$ be a normed vector space. A linear functional is a continuous linear map $\rho : X \to \mathbb{C}$.

Definition 1.10 (Dual space)**.** The dual space is the collection of all linear functionals defined by $\rho : X \to \mathbb{C}$, that is,

$$X^* = \{\rho : X \to \mathbb{C}, \rho \text{ continuous}\}.$$

The weak toplogy on X is the weakest topology on X such that $\rho : X \to \mathbb{C}$ is continuous for all $\rho \in X^*$. In other words, $\xi_n \to \xi$ in the weak topology if and only if

$$\rho(\xi_n) \to \rho(\xi) \text{ as } n \to \infty, \forall \rho \in X^*.$$

Pick $\eta \in \mathcal{H}$. Consider $\rho(\xi) = \langle \xi, \eta \rangle$.

Theorem 1.2 (Riesz representation theorem)**.** Any linear functional looks like this.

Proof. $\rho : \mathcal{H} \to \mathbb{C}$, consider $K = \ker \rho = \{\xi : \rho(\xi) = 0\}$. Look $K^\perp = \mathbb{C}\eta$. □

Consider $\{e_n\} \subset \ell^2(\mathbb{N})$, pick $\rho \in \ell^2(\mathbb{N})$. Consider

$$|\rho(e_n)| = |\langle e_n, \eta \rangle| = |\eta_n| \to 0 \implies e_n \to 0 \text{ weakly}.$$

1.4 Bounded Linear Operators

Definition 1.11 (Bounded linear operator)**.** Let us consider $B(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}, \text{ linear, continuous, w.r.t. norm topology}\}$.

Lemma 1.3. $T : \mathcal{H} \to \mathcal{H}$ is continuous at 0 if and only if it is bounded, i.e., $\sup\langle T\xi, T\xi \rangle < \infty, \|\xi\| = 1$.

Proof. Assume that T is continuous at $\xi = 0$. By linearity, we need to show that the image of some ball is bounded. $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\|T\xi\| < \epsilon, \forall \|\xi\| < \delta.$$

□

Definition 1.12. $\|T\| := \sup\limits_{\|\xi\|=1} \|T\xi\|$.

Definition 1.13. If $T_1, T_2 \in B(\mathcal{H})$, then $T_1 + T_2 \in B(\mathcal{H})$, $\lambda T_1 \in B(\mathcal{H})$ and $T_2T_1 \in B(\mathcal{H})$, $\|T_2T_1\| \le \|T_2\|\|T_1\|$ and $B(\mathcal{H})$ is a Banach space under $\|\cdot\|$. $B(\mathcal{H})$ is a Banach algebra, which is a normed algebra, complete and $\|ab\| \le \|a\|\|b\|\}$.

Example 1.3. In Hilbert space $\mathcal{H}$, $T : \mathcal{H} \to \mathcal{H}$, pick $\eta \in \mathcal{H}$, consider $\langle T\xi, \eta \rangle$, then $\xi \to \langle T\xi, \eta \rangle$ is a linear functional, then by the Riesz representation theorem, $\exists (T^*\eta) \in \mathcal{H}$, such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle, \forall \xi \in \mathcal{H}$.

Definition 1.14. $T^* : \mathcal{H} \to \mathcal{H}$ by $\eta \to T^*\eta$.

Lemma 1.4. $\|T^*\| = \|T\| = \|TT^*\|^{1/2}$.

1.5 C^*-Algebra

Theorem 1.3. $B(\mathcal{H})$ is a Banach * - algebra with the property $\|T\| = \|TT^*\|^{1/2}$ (C^*-equality), it is a C^*-algebra. Any norm closed *-subalgebra of $B(\mathcal{H})$ satisfies the same properties.

Definition 1.15. A concrete C^*-algebra is norm closed *-subalgebra of $B(\mathcal{H})$ for some $\mathcal{H}$.

Definition 1.16. An abstract C^*-algebra is a Banach *-algebra satisfying $\|T\|^2 = \|TT^*\|$.

Theorem 1.4. Any abstract C^*-algebra is isomorphic to a norm closed *-subalgebra of $B(\mathcal{H})$ for some $\mathcal{H}$.

Example 1.4. Let T be a group, then $\ell^2(\Gamma)$ is a Hilbert space. $\{e_\gamma, \gamma \in \Gamma\}$ is a basis of $\ell^2(T)$. $\forall \gamma \in \Gamma$, consider $e_{\gamma'} \to e_{\gamma'\gamma}, \gamma' \in \Gamma$. This induces an unitary on $\ell^2(\Gamma)$. $C^*_{\text{red}}(\Gamma) = \overline{\{\sum\limits_{\gamma \in \Gamma} c_\gamma u_\gamma\}}$.

Example 1.5. Let $T \in B(\mathcal{H})$.

- T is unitary if $T^*T = T^*T = 1$. We can conclude that $T^{-1} = T^*$, since

$$\begin{aligned} T^*T = 1 &\iff \langle T^*T\xi, \eta\rangle = \langle \xi, \eta\rangle, \forall \xi, \eta \in \mathcal{H} \\ &\iff \langle T\xi, T\eta\rangle = \langle \xi, \eta\rangle \\ &\iff \langle T\xi, T\xi\rangle = \langle \xi, \xi\rangle, \forall \xi \in \mathcal{H}. \end{aligned}$$

  Then T is an isometry.

Theorem 1.5. T is an isometry if $T^*T = 1$.

Theorem 1.6. In linear algebra, $T : \mathbb{C}^n \to \mathbb{C}^n$ if $\ker T = \{0\}$, then $R(T) = \mathbb{C}^n$ by rank and nullity, i.e., $\dim(\ker T) + \dim(R(T)) = n$. In Hilbert space, $\ker \mathrm{T} + \dim(R(T)) = \infty$.

Example 1.6. $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \{e_n, n \in \mathbb{Z}\}$.

(1) $T : e_n \to e_{2n}, n \in \mathbb{Z}$. T is isometric but not unitary. Then $T^* : e_{2n} \to e_n$ and $e_{2n+1} \to 0$.

(2) $T^*T(e_n) = T^*(e_{2n}) = e_n, \forall n \in \mathbb{Z} \implies T^*T = 1$.

(3) $TT^*(e_n)$. $TT^*(e_{2n}) = T(e_n) = e_{2n}$ and $TT^*(e_{2n+1}) = 0$.

(4) TT^* is the projection to $\mathrm{span}\{e_{2n}, n \in \mathbb{Z}\}$.

(5) $1 - TT^*$ is the projection to $\mathrm{span}\{e_{2n}, n \in \mathbb{Z}\}$.

Example 1.7. $\ell^2(\mathbb{N}) = \mathrm{span}\{e_1, e_2, \ldots, \}$.

(1) $T(e_n) = e_{n+1}, n = 1, 2, \ldots$, whereas $T^*(e_1) = 0$ and $T^*(e_{n+1}) = e_n, n = 1, 2, \ldots$. Then $T^*T = \mathbb{1}$ and $TT^* =$ project to $\mathrm{span}\{e_2, e_3, \ldots\}$. Then $\mathbb{1} - TT^* =$ projection to $\mathbb{C}e_1$.

(2) T^2 is shifting to right by 2. Then $(T^2)^*T^2 = \mathbb{1}$. $T^2(T^2)^* =$ projection to $\mathrm{span}\{e_3, e_4, \ldots\}$ and $\mathbb{1} - T^2(T^2)^* =$ projection to $\mathrm{span}\{e_1, e_2\}$.

Proposition 1.2. $0 \to \mathcal{K} \to C^*\{1, T\} \to C(\mathbb{T}) \to 0$.

Theorem 1.7.

(1) Unitary. $T^*T = TT^* = 1$.

(2) Isometry. $T^*T = 1$.

(3) Self-adjoint. $T = T^* \iff \langle T\xi, \xi\rangle \in \mathbb{R}$.

(4) Positive. $T = S^*S$ for some S. $\iff \langle T\xi, \xi\rangle \in \mathbb{R}^+$.

(5) Projection. $T = T^* = T^2$ if and only if T is the orthonormal projection to $T(\mathcal{H})$.

(6) Normal. $TT^* = T^*T \iff \langle T\xi, T\xi\rangle = \langle T^*\xi, T^*\xi\rangle$.

(7) The algebra generated by T and T^* ($^*-$ algebra) is commutative.

(8) v is a partial isometry if v^*v is a projection. (vv^* is a projection). Homework: Show that v^*v is a projection $\implies vv^*$ is a projection.

Definition 1.17. Let A be a unital algebra. Then $a \in A$, such that $\mathrm{sp}(a) = \{\lambda \in \mathbb{C}$ such that $a - \lambda I$ not invertable$\}$.

Theorem 1.8. If A is a Banach algebra, then $\mathrm{sp}(a) \neq \emptyset$ (compact).

Observation: If $a \in A$ with $\|a\| < 1$. Then we look at $1 + a + a^2 + \cdots + a^n + \cdots = (1 - a)^{-1} = \frac{1}{1-a}$

Corollary 1.2. $\mathrm{sp}(a) = \{\lambda \in \mathbb{C}, |\lambda| \leq \|a\|\}$.

Proposition 1.3. $\mathbb{C} \setminus \mathrm{sp}(a) \to \lambda \mapsto \frac{1}{a - \lambda I} \in A$ is holomorphic.

Theorem 1.9. Let A be a C^*-algebra, $aa^* = a^*a$, $C(\mathrm{sp}(a)) \cong C^*(1,a)$. $(z \mapsto z) \leftrightarrow a$, $f \leftrightarrow f(a)$, $z^2 \to a^2$, $f \cdot g \to f(a) \cdot g(a)$. Then $C^*(1,a)$ is called the continuous functional calculus.

Theorem 1.10 (Gelfand-Naimak)**.** If A is a unital commutative C^*-algebra, then $A = C^*(1,a) \approx C(\hat{A})$, where $\hat{A} = \{\rho : A \to \mathbb{C}, \rho \text{ are homomorphisms}\} \subset A'$ with point-wise convergence topology. That is, $\rho_n \to \rho \iff \rho(a_n) \to \rho(a), a \in A$. And A is a compact Hausdorff space. Note: $\ker \rho$ is a maximal ideal.

Proof. Let $A = C(X)$, $A' = (C(X))' = \{\mu : \rho_\mu : f \mapsto \int_X f d\mu\}$, $\mathrm{Prob}(X) = \{\mu : \mu(X) = 1\}$, which is a positive linear function norm. Then $\rho : f \mapsto \int f d\delta_X = f(x)$. □

The isomorphism is induced by the Gelfand map. $A \ni a \mapsto (\rho \mapsto \rho(a))$.

Proposition 1.4. $A = C^*(1,a) \cong C(\hat{A}) = C(\mathrm{sp}(a))$. Pick $\rho \in \hat{A}$, where $\rho : C^*(1,a) \to \mathbb{C}$. Then show $a - \rho(a) \cdot 1$ is not invertible. $\rho(a - \rho(a)1) = \rho(a) - \rho(\rho(a)1) = \rho(a) - \rho(a)\rho(1) = \rho(a) - \rho(a) = 0 \implies \rho(a) \in \mathrm{sp}(a)$.

Proposition 1.5.

(a) If $A = C(X)$ is separable, then X is ?

(b) If P is a projection in A, where $P = P^* = P^2$, then X is ?

Proposition 1.6. If X is a compact Hausdorff space, and $C(X) = \{f : X \to \mathbb{C}\}$ and $\|F\| = \sup\{|f(x)|, x \in X\}$. If $P = P^* = P^2 \in C(x)$, then $P(x) = \overline{P(x)} \implies P(x) \in \mathbb{R}$. $P(x) = P^2(x) \implies P(x) \in \{0,1\}$. That is to say, the projection in $C(X)$ corresponds to a clopen (closed and open) set of X. Ideal of $C(X)$ corresponds to closed subset of X: $I \leftrightarrow \{f : f|_Y = 0\}$.

Definition 1.18. Let A be a C^*-algebra, consider $A, M_2(A), M_3(A), \ldots, M_n(A)$, where

$$M_n(A) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

Then $A \leftrightarrow M_2(A) \leftrightarrow M_3(A) \leftrightarrow \cdots \leftrightarrow M_n(A)$, define $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$. Think about the projections in $M_\infty(A)$. Note: $M_\infty(A)$ is not a C^*-algebra.

Proposition 1.7. $A = C(X)$, $M_2(C(X)) = \left\{ \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, f_{ii} \in C(X), i = 1,2 \right\} = \{f : X \to M_2(\mathbb{C})\}$. If $P \in M_2(C(X))$, $P = P^* = P^2$, then $P(x) \in M_2(\mathbb{C})$ must be a projection.

Proposition 1.8. $x \mapsto P(x) \mapsto P(x)(\mathbb{C}^2) \subset \mathbb{C}^2$, where $x \in X$, $P(x) \in M_2(\mathbb{C})$. It is called the complex vector bundle over X. See projective module, vector bundle and Möbius strip.

Proposition 1.9. If $p, q \in M_\infty(A)$ and $p \sim q \iff \exists v \in M_\infty(A)$ such that $vv^* = p$ and $v^*v = q$.

Homework: reading assignment: projective module and vector bundle.

Proposition 1.10. Assume that A is a unital C^*-algebra. Consider $M_\infty(A) = \bigcup\limits_{n=1}^{\infty} M_n(A)$. A is a C^*-algebra, then the norm of A is coming from the following: Suppose that a is a matrix, then $\|a\|^2 = \sup\{\langle a\xi, a\xi\rangle, \|\xi\| = 1\} = \sup\{\bar{\xi}^T a^* a\xi, \|\xi\| = 1\} = \sup\{\sqrt{\lambda_i}, \lambda_i \in \mathrm{sp}(a^*a)\}$. Let $\mathrm{Proj}_\infty(A) = \{p : \text{projection of } M_\infty(A)\}$, p is Muray-von Neumann equivalent to q, denoted by $p \sim q$ if $\exists v \in M_\infty(A)$ such that $v^*v = p$ and $vv^* = q$, where $p, q \in \mathrm{Proj}_\infty(A)$.

Proposition 1.11. $\mathrm{Proj}_\infty(A)$ is a semi-group:

$$p \oplus q = \begin{bmatrix} p & \\ & q \end{bmatrix}, p \in M_m(A), q \in M_n(A), p \oplus q \in M_{m+n}(A).$$

Let $v(A) = \mathrm{Proj}_\infty(A)/\sim$, then $v(A)$ is an Abelian semi- group. Then

$$\begin{bmatrix} p & \\ & q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* \begin{bmatrix} q & \\ & p \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* \begin{bmatrix} q & \\ & p \end{bmatrix}^* \begin{bmatrix} q & \\ & p \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = w^* w.$$

Note that

$$ww^* = \begin{bmatrix} q & \\ & p \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* \begin{bmatrix} q & \\ & p \end{bmatrix}^* = \begin{bmatrix} q & \\ & p \end{bmatrix}^* \begin{bmatrix} q & \\ & p \end{bmatrix}^* = \begin{bmatrix} q & \\ & p \end{bmatrix}^*.$$

Definition 1.19 (unitary equivalence)**.** If there exists a unitary u ($u^*u = uu^* = 1$) such that $p = u^*qu$, then $p \sim_u q$.

Lemma 1.5. $p \sim_u q \implies p \sim q$.

Definition 1.20. $K_0(A)$ is the Grothediek group of $v(A)$.

Proposition 1.12. Take $A = \mathbb{C}$, then for $p \in M_n(\mathbb{A})$, then

$$p = u^* \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} u.$$

Given that

$$v(A) = \{0, \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}, \ldots\} = \mathbb{N} \cup \{0\}.$$

Then, $K_0(\mathbb{C}) = \mathbb{Z}$.

Proposition 1.13. $B(\mathcal{H}), \dim(\mathcal{H}) = c_0$, then $M_2(B(\mathcal{H})) \cong B(\mathcal{H}) \cong B(\mathcal{H} \oplus \mathcal{H})$. $p, q \in B(\mathcal{H})$ is a projection, consider $p(\mathcal{H}), q(\mathcal{H})$. $\exists v$ such that $v^*v = p, vv^* = q \iff \dim p(\mathcal{H}) = \dim q(\mathcal{H})$. If $v(B(\mathcal{H})) = \{0, 1, 2, 3, \cdots, n, \infty\}$, then $K_0(B(\mathcal{H})) = \{0\}$, note that $v(B(\mathcal{H}))$ is a semigroup.

Example 1.8. $v : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $e_n \to e_{n+1}$. It is known that $v^*v = 1$, $vv^* = p$, which is a projection to $\mathrm{span}\{e_2, e_3, \ldots\} \subsetneq \mathcal{H}$. Recall that $p \sim_u q$, if there exists a unitary such that

$$p = u^*qu.$$

Then $p \sim_u q \implies p \sim_{N,\ldots N} q$. If $1 \sim_u p$, there exists $u \in A$ which is a unitary such that $1 = u^*pu$. Then

$$1 = u1u^* = u(u^*pu)u^* = p.$$

Homework: Also, we have

$$p \sim_u q \iff p \sim q \;\&\; (1-p) \sim (1-q).$$

Lemma 1.6. If $p \sim q$, then

$$\begin{bmatrix} p & \\ & 0 \end{bmatrix} \sim_u \begin{bmatrix} q & \\ & 0 \end{bmatrix}.$$

Let $u = \begin{bmatrix} v & 1-q \\ 1-p & v^* \end{bmatrix}$, then

$$\begin{bmatrix} v & 1-q \\ 1-p & v^* \end{bmatrix} \begin{bmatrix} v^* & 1-p \\ 1-q & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} v^* & 1-p \\ 1-q & v \end{bmatrix} \begin{bmatrix} v & 1-q \\ 1-p & v^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} v & 1-q \\ 1-p & v^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^* & 1-p \\ 1-q & v \end{bmatrix} = \begin{bmatrix} vp & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^* & 1-p \\ 1-q & v \end{bmatrix} = \begin{bmatrix} vpv^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition 1.21. Given that $p \sim_u q$, if there exists a continuous function $h : [0,1] \to A$ such that

(a) $h(t)$ is a projection for all t.

(b) $h(0) = p$, $h(1) = q$.

Proposition 1.14.

$$p \sim_h q \implies p \sim_u q \implies p \sim q \implies \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \text{ and } p \sim_u q \implies \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \sim_h \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. Assume that $p \sim_u q$, then $p = u^* p u$, follows that

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}^* \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}.$$

Using the basic linear algebra trick, we have

$$\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u^* \end{bmatrix} \sim \begin{bmatrix} u^* & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^*.$$

□

Lemma 1.7. Let $p, q \in A$ be projection. If $\|p - q\| < \frac{1}{2}$, then $p \sim_u q$.

Proof. Consider $z = pq + (1-p)(1-q)$. Then

$$pz = ppq + p(1-p)(1-q) = pq.$$

and

$$zq = pqq + (1-p)(1-q)q = pq.$$

That implies that $pz = zq$. If z is invertible, then $p = zqz^{-1}$. Show that

$$\begin{aligned} \|z - 1\| &= \|pq + (1-p)(1-q) - 1\| \\ &= \|pq + 1 - p - q + pq - 1\| \\ &= \|pq - p + pq - q\| \\ &\le \|pq - p\| + \|pq - q\| \\ &= \|pq - p^2\| + \|pq - q^2\| \\ &= \|p(q-p)\| + \|(p-q)q\| \\ &\le \|p\|\|q-p\| + \|p-q\|\|q\| \\ &\le 2\|p-q\| < 1 \implies z \text{ is invertible.} \end{aligned}$$

□

Corollary 1.3. If A is separable, then $\mathrm{Proj}(A)/\sim_u$ is at most countable.

HOMEWORK: If $a, b \in A^+$, and $\|a\|, \|b\| \le 1$, then $\|a - b\| \le \|a\| + \|b\| \le 2$, but $\|a - b\| \le 1$. Note that $a \in A^+$ if and only if $a = c^*c$ for some c which is equivalent to $a = a^*$ and $\mathrm{sp}(a) \subset \mathbb{R}^+$.

Proposition 1.15.

$$\mathrm{Proj}_\infty(A)/\sim, \mathrm{Proj}_\infty(A)/\sim_u, \mathrm{Proj}_\infty(A)/\sim_h.$$

Proposition 1.16. Suppose that there exists a homotopy $\phi : A \to B$, then $[\phi] : K_0(A) \to K_0(B)$. If there is a continuous family of $\phi_t : A \to B$, where $t \in [0,1]$, then $t \mapsto \phi_t(a)$ is continuous. It follows that $\phi_0(p) \sim_h \phi_1(p) \implies [\phi_0] = [\phi_1]$.

Proposition 1.17. Provided that A is a unital C^*-algebra, then $A \rightsquigarrow V_0(A) \rightsquigarrow K_0(A)$. If $A = C(X)$ is unital, then X is compact. If X is not compact, then $C_0(X) = \{f : X \to \mathbb{C}, f(\infty) = 0\}$. Then $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}, f(-\infty) = f(\infty) = 0\}$, where $f(\infty) = 0 \iff \forall \varepsilon > 0, \exists K \subset X$ such that $|f(x)| < \varepsilon, x \in X \backslash K$, and K is compact.

Proposition 1.18. If X is NOT compact (connected), then $C_0(X) \otimes M_n$, does $C_0(X, M_n(\mathbb{C}))$ have nonzero projections?

Proof. Suppose that there exists a $P \in C_0(X, M_n(\mathbb{C}))$. Consider $\operatorname{tr}(P(x)) \in \mathbb{N}$, and consider the mapping $\phi : X \ni x \mapsto \operatorname{tr}(P(x)) \in \mathbb{N}$. Since ϕ is continuous, then $\operatorname{tr}(P(x))$ is a constant. Plus, $\lim_{x\to\infty} f(x) = 0$, therefore, $C_0(X, M_n(\mathbb{C}))$ does not have nonzero projections, that is, $P \in C_0(X, M_n(\mathbb{C})) \implies P = 0$, and $V_0(C_0(X)) = 0$. □

1.6 One Point Compactification

Definition 1.22. The compactification of $\mathbb{R}^1$ is a circle, and that of $\mathbb{R}^2$ is spherea, then that of $\mathbb{R}^n$ is S^n (n-sphere).

Definition 1.23 (Unification)**.** Let $A = C_0(X)$, then $\tilde{A} = A \oplus \mathbb{C}\mathbf{1}$. Then look at

$$(\alpha + \beta\mathbf{1})(\alpha' + \beta'\mathbf{1}) = (\alpha\alpha' + \alpha\beta' + \beta\alpha') + \beta\beta'.$$

The element of $\tilde{A}$ is $(a, \lambda), a \in A, \lambda \in \mathbb{C}$.

Proposition 1.19. If A, no matter unital or not, $K_0(A) := \ker([\pi] : K_0(\tilde{A}) \to K_0(\mathbb{C}))$, where $\pi : \tilde{A} \to \mathbb{C}, (a, \lambda) \mapsto \lambda$. $[P] \to [y], \pi_\infty(P) = r$, where $P \in M_\infty(\tilde{A}), r \in M_\infty(\mathbb{C})$.

Definition 1.24. Recall that $u \in A$ where A is a unital C^*-algebra is a unitary if $u^*u = uu^* = 1$. Consider the embeddings $U_n(A) \to U_{n+1}(A)$ by $u \mapsto \begin{bmatrix} u & \\ & 1 \end{bmatrix}$, where $U_n(A)$ is the unitary group of $M_n(A)$. Hence $U_\infty(A) = \bigcup_n^\infty U_n(A)$ is a topological group. Assume that $(U_\infty)_0 = \{u : u \text{ is path connected to } 1\}$, then $K_1(A) = U_\infty(A)/(U_\infty)_0$.

Lemma 1.8. If u, v are unitaries in A, $[u]_1$ (K_1 class of u). Then

$$[u \cdot v]_1 = [v \cdot u]_1 = [u]_1 + [v]_1 = \left[\begin{bmatrix} u & \\ & v \end{bmatrix}\right]_1.$$

Also,

$$uv = \begin{bmatrix} uv & \\ & 1 \end{bmatrix} = \begin{bmatrix} u & \\ & 1 \end{bmatrix} \begin{bmatrix} v & \\ & 1 \end{bmatrix}.$$

Then use the trick, let

$$u_\theta = \begin{bmatrix} u & \\ & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} v & \\ & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{bmatrix}^*, 0 \le \theta \le \frac{\pi}{2}.$$

Then

Lemma 1.9. If $\|u - v\| < 2$, then $u \sim v$ in A.
Sketch of proof: Show that $K_1(\mathbb{C}) = \{0\}$. Let's show that if $u \in M_n(\mathbb{C})$, $uu^* = u^*u = 1$, then $u \sim 1_{n\times n}$.

$$u = v \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} v^* = v \begin{bmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{bmatrix} v^*$$

Note that $|\lambda_k| = 1 \implies \lambda_k = e^{i\theta_k}, \theta_k \in [0, 2\pi]$.

Proof. Since $u = (uv^*)v$, consider uv^*, which is a unitary,

$$\|uv^* - 1\| = \|(u - v)v^*\| < 2.$$

We can conclude that $\mathrm{sp}(uv^*) \in \mathbb{T}$, where $\mathbb{T}$ is unit circle. Since $\|uv^* - 1\| < 2$, then $\mathrm{sp}(uv^*) \neq \mathbb{T}$. Then $C^*(uv^*) = C(\mathrm{sp}(uv^*))$. So, $uv^* \leftrightarrow z \to z$, where $z = e^{i\theta}, -\pi < \theta < \pi$. Then $\exists h \in A$ that $h = h^*$ such that $uv^* = e^{ih}$. Consider $e^{ih(1-t)} \cdot v$, then we have $u_0 = u$ and $u_1 = v$. ☐

Example 1.9. (a) $K_1(\mathbb{C}) = \{0\}$.

(b) $K_1(B(\mathcal{H})) = \{0\}$.

(c) In $B(\mathcal{H})$, any unitary $u = e^{iH}$, where $H = H^*$.

(d) $C(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, f \text{ continuous}\}$. Then $u \in U_n(C(\mathbb{T})) = C(\mathbb{T} \to U_n(\mathbb{C}))$. Consider $\mathbb{T} \ni z \implies \det(u(z)) \in \mathbb{T}$.

Definition 1.25. Setting: A is a C^*-algebra, Γ is a locally compact group, and $\alpha : \Gamma \to \mathrm{Aut}(A)$ which is a homomorphism. Then $g \mapsto \alpha_g \in \mathrm{Aut}(A)$. Then $\forall a \in a$, $\Gamma \ni g \mapsto \alpha_g(a) \in A$. It follows that (A, Γ, α) is called a C^*-dynamical system. Then

$$A \rtimes_\alpha \Gamma = \overline{C_c(\Gamma, A)^{\|\cdot\|}}, f * g(\gamma) = \int_\Gamma f(r\gamma_1)g(\gamma_1^{-1})\, d\gamma_1.$$

If Γ is discrete, then

$$A \rtimes_\alpha \Gamma = C^*\{A, U_\gamma : U_\gamma \text{ unitaries such that } U_{\gamma_1}U_{\gamma_2} = U_{\gamma_1\gamma_2}, U_\gamma^* a U_\gamma = \alpha_\gamma(a)\}.$$

Proposition 1.20. Let $A = C(\mathbb{T})$, $\alpha \in \mathrm{Aut}(C(\mathbb{T}))$, $\alpha_\theta(f)(z) = f(e^{2\pi i\theta}z)$. Consider $A_\theta = C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z} = C^*\{C(\mathbb{T}), v : v^* f(\cdot) v = f(e^{2\pi i\theta}\cdot)\} = C^*(z \mapsto z) = C^*(u) = C^*\{u, v : v^*uv = \alpha(u) = e^{2\pi i\theta}u \implies uv = e^{2\pi i\theta}vu\}$.

- If $\theta = 0$, then $f = \sum_{m,n} C_{mn}u^m v^n = C^*(u) \otimes C^*(v) = C(\mathbb{T}) \otimes C(\mathbb{T}) = C(\mathbb{T} \times \mathbb{T})$. The cartesian product of two circles are torus.
- If $\theta \notin \mathbb{Q}$, then A_θ is simple. $K_0(A_\theta) \cong \mathbb{Z}^2$ and $K_0(A_\theta) \cong \mathbb{Z}^2$.

HOMEWORK: Check the ideal generated by M_n over $\mathbb{C}$ is simple. Presentation: Something about rotation algebra, show that non-rotation algebra is simple, and trace things like that.

Definition 1.26. For an arbitrary ring $(R, +, \cdot)$, let $(R, +)$ be its additive group. A subset I is called a two-sided ideal of R if it is an additive subgroup of R that "absorbs multiplication by elements of R". Formally we mean that I is an ideal if it satisfies the following conditions:

(1) $(I, +)$ is a subgroup of $(R, +)$.

(2) $\forall x \in I, \forall r \in R : x \cdot r, r \cdot x \in I$.

Proposition 1.21. Let $A_\theta = \{u, v : uv = e^{2\pi i\theta}vu\}$, and we have the self-adjoint operator $H_\theta = u+u^*+v+v^*$.

(1) $\mathrm{sp}(H_\theta)$ is a Cantor set for all $\theta \in \mathbb{Q}$.

(2) All gaps are open. That is to say, $[f_t(H)]_0 \in K_0(A_\theta)$.

Proposition 1.22. $0 \to I \to A \to J \to 0$ is exact, then we apply the K_* functor, where $* = 0, 1$. Then

$$0 = K_*(0) \to K_*(I) \to K_*(A) \to K_*(J) \to K_*(0) = 0,$$

in general, $K_*(I)$ and $K_*(J)$ is not exact, while $K_*(A)$ is exact. However, we have the exact map (six-term exact sequence):

$$K_1(I) \to K_1(A) \to K_1(J) \to K_0(I) \to K_0(A) \to K_0(J) \to K_1(I),$$

where $K_1(J) \to K_0(I)$ is called the index map and $K_0(J) \to K_1(I)$ is called the exponential map.

Definition 1.27. Let A be a C^*-algebra, then I is an (closed and two-sided) ideal of A. It follows that $A/I = \{a + I, a \in A\}$ and define $\|a + I\| := \inf\{\|a + b\|, b \in I\}$. We use the exact sequence, $0 \to I \to A \to A/I \to 0$.

Example 1.10. Let $\mathcal{H} = \ell^2(\mathbb{N})$, $T \in B(\mathcal{H})$ is compact if $\overline{T(B)}$ is compact in $\|\cdot\|$, where B is the unit ball of $\mathcal{H}$.

Theorem 1.11. $T \in B(\mathcal{H})$ is compact if and only if it can be approximated by a finite rank operator, i.e., $\exists T_n$ such that $\|T_n - T\| \to 0$ and $\dim(T(\mathcal{H})) < \infty$.

Example 1.11.

$$T = \begin{bmatrix} 1 & & & & & \\ & \frac{1}{2} & & & & \\ & & \frac{1}{3} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{n} & \\ & & & & & \ddots \end{bmatrix} \text{ can be approximated by } T' = \begin{bmatrix} 1 & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ & & & \ddots & \\ & & & & \frac{1}{n} \end{bmatrix}.$$

Proposition 1.23. $M_n(\mathbb{C}) \mapsto M_{n+1}(\mathbb{C}) \mapsto \cdots \mapsto M_\infty(\mathbb{C})$. Denote by $\mathcal{K}$ the collection of all compact operators. Conclude that $\mathcal{K}$ is a closed and two-sided ideal, hence we have $0 \to \mathcal{K} \to B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K} \to 0$. $B(\mathcal{H})/\mathcal{K}$ is called the Calkin algebra.

Proposition 1.24. With $\mathcal{H} = \ell^2((N))$ if there is a family of mutually orthogonal projections (sum to 1) in $B(\mathcal{H})$, then this family is at most countable. Note that mutually orthogonal means $\{P_\lambda, \lambda \in \Lambda\}$, and $P_{\lambda_1} \cdot P_{\lambda_2} = 0$ if $\lambda_1 \neq \lambda_2$; sum 1 means

$$1 = \begin{bmatrix} P_1 & & & & \\ & P_2 & & & \\ & & \ddots & & \\ & & & P_n & \\ & & & & \ddots \end{bmatrix}.$$

In $B(\mathcal{H})/\mathcal{K}$, however, it's possible to find uncountable family of mutually orthogonal projections. Plus, if $B(\mathcal{H})/\mathcal{K} \leftrightarrow B(\mathcal{H}')$, then $\mathcal{H}'$ cannot be separable.

Theorem 1.12. $\mathcal{K}$ is the only non-trivial ideal of $B(\mathcal{H})$.

Corollary 1.4. $B(\mathcal{H})/\mathcal{K}$ is simple (no non-trivial closed and two-sided ideal).

Corollary 1.5. $\mathcal{K}$ is also simple. (no non-trivial closed two-sided ideal). $\bigcup_{n=1}^\infty M_n \lhd \mathcal{K}$.

Question:

1. Is $B(\mathcal{H})/\mathcal{K}$ algebraically simple?
2. If $1 \in A$ is topologically simple, is A algebraically simple?

Proof. Proof by contradiction. Suppose that $J \lhd A$ is an algebraic ideal. Then $\overline{J} \lhd A$. Since A is simple, $\overline{J} = A \ni 1$ and show that $1 \in J$. Hint: spectrum... □

Question: $K_*(B(\mathcal{H})/\mathcal{K}) =?$, where $* = 0, 1$. Then we'll have index map and Fredholm operator.

Example 1.12. Given that $0 \to I \to A \to J \to 0$. If $A = C_0((0,1])$, $I = C_0((0,1))$, $J = \mathbb{C}$, then

$$0 \to C_0((0,1)) \to C_0((0,1]) \to \mathbb{C} \to 0.$$

It turns out that $K_*(A) = \{0\}$, where $* = 0, 1$. $\tilde{I} = C(\mathbb{T})$ and $K_1(I) = K_1(\tilde{I}) = \mathbb{Z}$, and $K_0(\tilde{I}) = \mathbb{Z} = \{-2, -1, 0, \ldots\}$, $K_0(I) = \{0\}$.

Definition 1.28. $0 \to I \to A \to J \to 0$ splits if $\exists \eta : J \to A$ such that $\pi\eta = 1_J$.

Example 1.13. $0 \to C_0((0,1]) \to C([0,1]) \to \mathbb{C} \to 0$, where $\pi : C([0,1]) \to \mathbb{C}$ and $\eta : \lambda \mapsto (t \mapsto \lambda)$.

Definition 1.29. Lifting of elements: $A \to J \to 0$. For example,

1. The self-adjoint element: $\pi(c) = a$ and $\pi(c^*) = a^*$, since $a = a^*$, that is, $\pi(c) = \pi(c^*)$. Then create an element $\tilde{c} = (c + c^*)/2$.

2. The positive element $(a = bb^*)$. Then $c \mapsto a$ and $c^* \mapsto a^*$, so $cc^* \mapsto a$.

3. Can we pick c such that $\|c\| = a$? Then $a = f(a)$ and $\|f(c)\| = \|a\|$. Also, $\pi(f(c)) = f(\pi(c)) = f(a) = a$. Given that $\pi : A \to J, c \mapsto a$ then $\mathrm{sp}(c) \supset \mathrm{sp}(a)$. Show the last statement:

$$\{\lambda \in \mathbb{C} : \lambda I - c \text{ is invertible}\} = (\mathrm{sp}(c))^{\complement} \subset (\mathrm{sp}(a))^{\complement} = \{\lambda \in \mathbb{C} : \lambda I - a \text{ is invertible}\}.$$

   Then $\exists b, d \in A$, then $(\lambda I - c)b = d(\lambda I - c) = 1$ and $(\lambda I - a)\pi(b) = \pi(d)(\lambda I - a) = 1$.

Proposition 1.25. Let $\sigma : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, i.e., $e_n \mapsto e_{n+1}$, then $\sigma^*\sigma = 1$ and $\sigma\sigma^* =$ projection onto $\operatorname{span}\{e_2, e_3, \ldots\}$. It follows that $1 - \sigma\sigma^* =$ projection to $\mathbb{C}e_1$ which is a rank one projection. Now consider $(\sigma^2)^*\sigma^2 = 1$ and $\sigma^2(\sigma^2)^* =$ projection onto $\operatorname{span}\{e_3, e_4, \ldots\}$, $1 - \sigma^2(\sigma^2)^* =$ projection to $\operatorname{span}\{e_1, e_2\}$. As a consequence, we have

$$0 \to \mathcal{K} \to C^*(\sigma, 1) \to C^*(\pi(\sigma), 1) \to 0,$$

where $\pi(\sigma)\pi(\sigma^*) = \pi(\sigma^*)\pi(\sigma) = 1$.

Definition 1.30. Let $T : \mathcal{H} \to \mathcal{H}$ is Fredholm if

$$\dim(\ker T) < \infty, \dim(\text{co-ker}\, T) = \dim(\ker T^*) < \infty.$$

Then the index of T is defined by

$$\operatorname{Ind} T := \dim(\ker T^*) - \dim(\ker T).$$

Theorem 1.13. T is Fredholm if and only if $\pi(T)$ is invertible, where $\pi : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}$.

Theorem 1.14. If T is Fredholm, then $T + k$ is Fredholm for any $k \in \mathcal{K}$.

Theorem 1.15. The map $T \to \operatorname{Ind} T$ is continuous.

Corollary 1.6. If there is a continuous path of the Fredholm operator $T_t, t \in [0,1]$, then $\operatorname{Ind} T_0 = \operatorname{Ind} T_1$, in particular, $\operatorname{Ind}(T + k) = \operatorname{Ind} T$.

Proposition 1.26. Consider $A \to J \to 0$, if $u \in U_0(J)$, where $U_0(J)$ is the connected component of $U(J)$ containing 1, then u can be lifted. If $u = e^{ih}$, then $u \in U_0$, where h is self-adjoint. If $h \to h'$, then $u = e^{ih} \to u' = e^{ih'}$. Also, $u = e^{ih_1}e^{ih_2} \cdots e^{ih_n}$ can be lifted.

Lemma 1.10. $U_0(A) = \{e^{ih_1}e^{ih_2} \cdots e^{ih_n}, h_i \in A \text{ s.a.}, i = 1, 2, \ldots, n\}$.

Proof. HINT: Show that $G = e^{ih_1}e^{ih_2} \cdots e^{ih_n}$ is closed and open. Use the following trick: If $\|u - 1\| < 2$, then $u = e^{ih}$. □

Example 1.14. Let X be a cantor set. And : $X \to X$ is a homeomorphism. $C(X) = \{f : X \to \mathbb{C}\}$ and $\sigma : f \mapsto f \circ \sigma^{-1}$ and $C(X) \rtimes \mathbb{Z} = \{\sum_{n\in\mathbb{Z}} f_n \cdot u^n, u^*fu = \sigma(f)\}^{\|\cdot\|}$. $(f_1u^n)(f_2u^m) = f_1u^nf_2u^m = f_1\sigma^nf_2u^nu^m = f_1\sigma^nf_2u^{n+m}$. Assume that we have closed subset $Y \subset X$, such that $(Y) = Y$, then we have $0 \to C_0(X\backslash Y) \to C(X) \to C(Y) \to 0$, follows that $0 \to C_0(X\backslash Y) \rtimes \mathbb{Z} \to C(X) \rtimes \mathbb{Z} \to C(Y) \rtimes \mathbb{Z} \to 0$.

Example 1.15.

1. $K_0(C(X)) = C(X, \mathbb{Z})$.

2. $K_1(C(X)) = \{0\}$.

3. $K_0(C(X) \rtimes \mathbb{Z}) = C(X, \mathbb{Z})/\{f - f \circ \sigma^{-1}$.
4. $K_1(C(X) \rtimes \mathbb{Z}) = [u]_1$.

Definition 1.31. $E \subset X$ is clopen, χ_E is a projection.

Example 1.16. $[\chi_E] \iff \sigma(\chi_E) = [u\chi_E u^*]$.

Example 1.17. $X = \infty \cup \mathbb{Z}$, and $\sigma : \infty \mapsto \infty, n \mapsto n+1$. Then

$$0 \to \mathbb{Z} \rtimes_\sigma \mathbb{Z} \to A \to \mathbb{C} \rtimes_\sigma \mathbb{Z} = C(\mathbb{T}) \to 0.$$

Proposition 1.27.

$$\sum_{n \in \mathbb{Z}} c_n u^n = C^*(u) \cong C(\mathrm{sp}(u)) = C(\mathbb{T}).$$

Proposition 1.28. If $C^*(u)$ is a universal C^*-algebra generated by a unitary. $C^u, \exists \varphi$ such that $u \to v$, then $C^*(v) \subset A$. Hence we have

$$\forall z \in \mathbb{T}, C^*(u) \to^{\varphi_z} C^*(zu) = C^*(u).$$

In other words, the circle action $\mathbb{T} \ni z \mapsto \varphi_z \in \mathrm{Aut}(C^*(u))$

Homework: p, q, v, we have $p^2 = p = p^*$, $q^2 = q = q^*$, and $vv^* = p, v^*v = q$ and $p + q = 1$. Then show that the C^*-algebra generated by p, q, v is $M_2(\mathbb{C})$.

Proposition 1.29. Let $Y \leftrightarrow X$, and $\sigma : X \to X$, and $\sigma(Y) = Y$. Then $0 \to C_0(X \backslash Y) \rtimes \mathbb{Z} \to C(X) \rtimes \mathbb{Z} \to C(Y) \rtimes \mathbb{Z} \to 0$.

Example 1.18. $X = \mathbb{Z} \cup \{\infty\}$ and $Y = \{\infty\}$, then $0 \to C_0(\mathbb{Z}) \rtimes \mathbb{Z} \to C(X) \rtimes \mathbb{Z} \to C(\infty) \rtimes \to 0$, where $C_0(\mathbb{Z}) \rtimes \mathbb{Z} \cong \mathcal{K}$ and $C(\infty) \rtimes \mathbb{Z} \cong C(\mathbb{T})$. Take N to be a neighborhood at ∞, $\chi_N \in C(X)$, $\pi(\chi_N) = 1$, and $\chi_N u \in C(X) \rtimes (Z)$ and $\pi(\chi_N u) = u$. And then we have

$$(u^*\chi_N)(\chi_N u) = u^*\chi_N u = [M+1, -M+1], (\chi_N u)(u^*\chi_N) = \chi_N uu^*\chi_N = \chi_N = [M, -M].$$

Then $\mathrm{Ind}(u) = [M+1, M] - [-M+1, -M] = [\chi_N \circ \sigma - \chi_N]$.

Example 1.19. $X = \mathbb{Z} \cup \{\pm\infty\}$, and $Y = \{\pm\infty\}$. Consider $\chi_{[-\infty, N]} - \chi_{[-\infty, N+1]} = -\chi_{\{N+1\}}$.

Definition 1.32. $0 \to I \to A \to J \to 0$, consider $[u] \in K_1(J)$, if u can be lifted to a partial unitary $v \in A$, consider $I - v^*v, I - vv^* \in I$. Then

$$\mathrm{Ind}[u]_1 = [I - v^*v]_0 - [I - vv^*]_0.$$

Example 1.20. If $u \curvearrowright a$, where $\|a\| \le 1$ and $a \curvearrowright b$ means a can be lifted to b, and

$$\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \curvearrowright \begin{bmatrix} a & 0 \\ \sqrt{1 - a^*a} & 0 \end{bmatrix}$$

Individual Project Topic Pools:

1. C^*-algebra, dynamical system, and classification.
2. Rotation algebra $A_\theta = \{u, v : uv = e^{2\pi i\theta} vu, u, v \text{ are unitaries}\}, \theta \in [0, 1]$. If $\theta \notin \mathbb{Q}$, then A_θ is simple and there is a unique trace. $\tau(K_0(A_\theta)) = \mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$. Higher dimensional non-commutative tori. (Exercise 5.8 in textbook)
3. Cuntz algebra, $n = 2, 3, \ldots, \infty$. Then $O_2 = C^*\{v_1, v_2 : v_1^*v_1 = 1, v_2^*v_2 = 1, v_1v_1^* + v_2v_2^* = 1\}$. O_2 is simple and it has Lie group action (T 2 action). $K_*(O_2) = \{0\}$. Then $O_2 \otimes O_2 = O_2$, $O_n \otimes O_2 = O_2$, $M_n \otimes O_2 = O_2$ and $A_\theta \otimes O_2 = O_2$, especially $A \otimes O_\infty = A$ and $K_0(O_n) = \mathbb{Z}/(n-1)\mathbb{Z}$.
4. Expanders, k-groups and graphs (require advanced algebraic topology).

5. Projective module, Swan's theorem.
6. Classification of CCR algebras.
7. Classification of $O_2 \rtimes \mathbb{R}$ (cross product algebra).
8. Graph C^*-algebras.
9. C^*-algebra in numerical analysis.
10. The (strict) comparison of projections by traces (regularity of C^*-algebra non-commutative topological dimension)

Proposition 1.30. If $\tau \in A'$ is a trace if $\tau(ab) = \tau(ba)$. Then $[\tau] : K_0(A) \to \mathbb{R}$ and $[P] \mapsto \tau(P)$. It follows that

$$K_0(A) \to AH, S(K_0(A)) := \{P : K_0(A) \to \mathbb{R}, \rho[1] = 1\}.$$

1.7 Irrational Rotation Algebra

Definition 1.33. Let $\theta \in [0,1]\backslash\mathbb{Q}$, then $A_\theta := C^*\{u, v : uv = e^{2\pi i\theta}vu, u, v \text{ are unitaries}\}$.

Theorem 1.16. A_θ is simple and has a unique trace.

Proposition 1.31. Let us take a look at poly$[u,v]/uv = e^{2\pi i\theta}vu$. Fix $\mathcal{H}$ separable Hilbert space. Consider $\Lambda = \{(u,v), uv = e^{2\pi i\theta vu}, u, v \text{ are unitaries}\}$. Define the operator norm

$$\|f\| = \sup\{\|f(u,v)\|, (u,v) \in \Lambda\} \leq \|f\|_1 = \sum_{m,n} |c_{m,n}| < \infty,$$

where $f = \sum_{m,n} u^m v^n$.

Example 1.21. $C^*\{a : a = a^*\}$ does NOT exist.

1.8 Trace on A_θ

Definition 1.34. The operator τ is said to be a trace if $\tau : A_\theta \to \mathbb{C}$, $\tau(1) = 1$, $\tau(ab) = \tau(ba)$ and $\tau(aa^*) \geq 0$,

Proposition 1.32. Suppose that $f \in C(\mathbb{T})$, if $f = c_{-n}z^{-n} + \cdots + c_{-1}z^{-1} + c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ and μ is a Lebesgue measure on the unit circle $\mathbb{T}$ ($\mu(\mathbb{T}) = 1$), then

$$\mu(f) = \int f d\mu = c_{-n}\int z^{-n} + \cdots + c_{-1}\int z^{-1} + c_0 + c_1\int z + c_2\int z^2 + \cdots + c_n\int z^n = c_0.$$

Note: $\int z = \int_0^1 e^{2\pi i\theta} d\theta = 0$. Look for a $\tau : A_\theta \to \mathbb{C}$ such that $\tau(u^m v^n) = 0$ if $|m| + |n| \neq 0$. Consider $A_\theta = C^*(u,v)$, then $z_1 u, z_2 v \in A_\theta$ where $z_1, z_2 \in \mathbb{T}$. Assume there exists $\sigma_{z_1,z_2} : A_\theta \to A_\theta$, $\sigma_{z_1,z_2} : (u,v) \mapsto (z_1 u, z_2 v)$. Then σ_{z_1,z_2} is group action $\mathbb{T} \times \mathbb{T} \curvearrowright A_\theta$. Then $z_1, z_2 \in \mathbb{T}$, $a \in A_\theta$, then we can construct the trace $\tau(a) = \int_{\mathbb{T}\times\mathbb{T}} \sigma_{z_1,z_2}(a) d(z_1,z_2)$.

Proof. Consider $\sigma_{z_1,z_2}(u^m v^n) = z_1^m u^m z_2^n v^n = z_1^m z_2^n u^m v^n$, then

$$\int_{\mathbb{T}\times\mathbb{T}} \sigma_{z_1,z_2}(u^m v^n) dz_1 z_2 = \int_{\mathbb{T}\times\mathbb{T}} z_1^m z_2^n u^m v^n dz_1 z_2 = \int_{\mathbb{T}} z_1^m dz_1 \int_{\mathbb{T}} z_2^n dz_2 u^m v^n = \sigma_{0,n}\sigma_{m,0}\mathbf{1}.$$

□

HOMEWORK: prove that $\tau(ab) = \tau(ba)$. If τ' is a trace on A_θ, then $\tau'(uv) = \tau'(vu) = \tau'(e^{2\pi i\theta}vu) = e^{2\pi i\theta}\tau'(vu)$, since θ is irrational, that implies that $\tau'(uv) = 0$, i.e., the trace we constructed is unique.

Proposition 1.33. Gauge action: $\mathbb{T}^2 \curvearrowright A_\theta$, $(u,v) \mapsto (z_1 u, z_2 v)$. The trace operator $\tau(a) = \int_{\mathbb{T}^2} \sigma_{z_1,z_2}(a)\, dz_1 z_2$ (looking at the orbit of all elements).

Proposition 1.34. Let $A_\theta := C^*\{u, v : uv = e^{2\pi i\theta}vu, u, v \text{ are unitaries}\}$, $a \in A_\theta$, then

$$\Phi_{N,u}(a) = \frac{1}{N}(uau^{-1} + u^2au^{-2} + u^3au^{-3} + \cdots + u^Naa^{-N}).$$

If $a = u$, then

$$\Phi_{N,u}(u) = \frac{1}{N}(uuu^{-1} + u^2uu^{-2} + u^3uu^{-3} + \cdots + u^Nuu^{-N}) = u.$$

If $a = u^m$, then

$$\Phi_{N,u}(u^m) = u^m.$$

If $a = v$, then

$$\begin{aligned}
\Phi_{N,u}(v) &= \frac{1}{N}(uvu^{-1} + u^2vu^{-2} + u^3vu^{-3} + \cdots + u^Nvv^{-N}) \\
&= \frac{1}{N}(e^{2\pi i\theta}v + e^{2\pi i2\theta}v + \cdots + e^{2\pi iN\theta}v) \\
&= \frac{1}{N}(e^{2\pi i\theta} + e^{2\pi i2\theta} + \cdots + e^{2\pi iN\theta})v \\
&= \frac{1}{N}[e^{2\pi i\theta} + (e^{2\pi i\theta})^2 + \cdots + (e^{2\pi i\theta})^N]v \\
&= \frac{1}{N}\frac{1-(e^{2\pi i\theta})^N}{1-e^{2\pi i\theta}}v \to 0 \text{ as } N \to \infty.
\end{aligned}$$

If $a = v^m$, then

$$\begin{aligned}
\Phi_{N,u}(v^m) &= \frac{1}{N}(uv^mu^{-1} + u^2v^mu^{-2} + u^3v^mu^{-3} + \cdots + u^Nv^mv^{m-N}) \\
&= \frac{1}{N}[e^{2\pi im\theta} + (e^{2\pi im\theta})^2 + \cdots + (e^{2\pi im\theta})^N]v^m \\
&= \frac{1}{N}\frac{1-(e^{2\pi im\theta})^N}{1-e^{2\pi im\theta}}v^m \to 0 \text{ as } N \to \infty.
\end{aligned}$$

If $a = u^mv^n$, then

$$u(u^mv^n)u^* = u^{m+1}v^nu^* = u^m(uv^nu^*) = (e^{2\pi in\theta})u^mv^n,$$

it follows that

$$\Phi_{N,u}(a) \to 0 \text{ as } N \to \infty.$$

Definition 1.35. $\Phi_u(a) := \lim_{N\to\infty} \Phi_{N,u}(a) \in C^*(u)$. Similarly, $\Phi_v(a) := \lim_{N\to\infty} \Phi_{N,v}(a) \in C^*(v)$. Then

$$\Phi(a) = \Phi_u(a) \circ \Phi_v(a) = \Phi_v(a) \circ \Phi_u(a) = \lim_{M,N\to\infty} \frac{1}{(2M+1)(2N+1)} \sum_{|m|\le M,|n|\le N} u^mv^nav^{-n}u^{-m}.$$

It follows that $\Phi(a) \in \mathbb{C}1$ and we have $\Phi(a) = \tau(a)1$. Note that

$$\lim_{M,N\to\infty} \frac{1}{(2M+1)(2N+1)} \sum_{|m|\le M,|n|\le N} u^mv^nav^{-n}u^{-m} = \tau(a)1.$$

If ρ is a trace, then

$$\lim_{M,N\to\infty} \frac{1}{(2M+1)(2N+1)} \sum_{|m|\le M,|n|\le N} \rho(u^mv^nav^{-n}u^{-m}) = \rho(\tau(a)1) = \rho(\tau(a))\rho(1) = \tau(a).$$

If the ideal $J \lhd A_\theta$, $a \in J$, then

$$\Phi(a) = \lim_{M,N\to\infty} \frac{1}{(2M+1)(2N+1)} \sum_{|m|\le M,|n|\le N} u^mv^nav^{-n}u^{-m} = \tau(a)1 \in J.$$

If $\tau(a) \neq 0$, then $J = A_\theta$.

Lemma 1.11. The trace τ is said to be *faithful* if $\tau(aa^*) \gneqq 0$ if $a \neq 0$.

Proof. Since $\sigma_{z_1 z_2}(aa^*) > 0$, then

$$\tau(aa^*) = \int_{\mathbb{T}^2} \sigma_{z_1 z_2}(aa^*)\, dz_1 z_2 \neq 0.$$

Then we can conclude that A_θ is simple. $\square$

Proposition 1.35. $K_0(A_\theta) = \mathbb{Z}^2$, and $K_1(A_\theta) = \mathbb{Z}^2$.

Lemma 1.12. There is a projection $P \in A_\theta$ such that $\tau(P) = \theta$. It is so-called Rieffle projection.

HOMEWORK: Find out f, g, h such that

$$P := f(u)v^* + g(u) + h(u)v,$$

where $f : \mathbb{T} = \mathrm{sp}(u) \to \mathbb{C}$, is (approximately) a projection, i.e., $P^2 \approx P$, $P = P^*$ and $\int g = \theta$. Note that $vf(u)v^* = f(e^{2\pi i\theta}u)$.

Proof. First show that $P^2 \approx P$ as follows, let $P = g(u)v^* + f(u) + vg(u) \in A_\theta$, where $f, g \in C^+(\mathbb{T})$, and we have $vhv^* = \sigma(h) \implies hv^* = v^*\sigma(h) \implies v^*\sigma^{-1}(h) = h^{-1}v^*$.

$$\begin{aligned}[g(u)v^* + f(u) + vg(u)]^2 &= g\sigma^{-1}(g)(v^*)^2 + g\sigma^{-1}(f)v^* + g^2 + fgv^* + f^2 + f\sigma(g)v + \sigma(g^2) + \sigma(g)\sigma(f)v + \sigma(g)\sigma^2(g)v^2 \\ &= gv^* + f + vg.\end{aligned}$$

Then we get following equations

$$\begin{cases} g\sigma^{-1}(g) \approx 0, g\sigma(g) \approx 0 \\ g\sigma^{-1}(f) + fg = g \\ f\sigma(g) + \sigma(g)\sigma(f) = \sigma(g) \\ g^2 + \sigma(g^2) + f^2 = f. \end{cases}$$

$\square$

QUESTION: When will $A_\theta \cong A_{\theta'}$? If and only if $\theta = \pm\theta'$.

Lemma 1.13. There is a $P \in A_\theta$ such that $\tau(P) = \theta$.

Lemma 1.14. $K_0(A_\theta) \cong \mathbb{Z}^2 = \langle [1], [P] \rangle$, $K_1(A_\theta) \cong \mathbb{Z}^2 = \langle [u], [v] \rangle$.

Corollary 1.7. $\tau(K_0(A_\theta)) = m1 + n\theta \subset \mathbb{R}, m, n \in \mathbb{Z}$.

Proposition 1.36. If $A_\theta \cong A_{\theta'}$, then $\mathbb{Z} + \mathbb{Z}\theta = \mathbb{Z} + \mathbb{Z}\theta' \iff \theta = \theta'$ or $\theta = 1 - \theta'$.

Proof. $A_{1-\theta} = \{u, v : uv = e^{2\pi i(1-\theta)}vu\} \implies \{u, v : vu = e^{2\pi i\theta}uv\}$. $\square$

1.9 Pimsner-Voiculescu Six-term Exact Sequence

QUESTION: Given the C^*-algebra A, we have $\alpha \in \mathrm{Aut}(A)$, how to calculate the following

$$K_*(A \rtimes_\alpha \mathbb{Z}).$$

SOLUTION. We should have the following exact sequence,

$$K_0(A) \overset{1-[\alpha]_0}{\to} K_0(A) \overset{[i]_0}{\to} K_0(A \rtimes \mathbb{Z}) \to K_1(A) \overset{1-[\alpha]_1}{\to} K_1(A) \overset{[i]_0}{\to} K_1 A \rtimes \mathbb{Z} \to K_0(A).$$

Recall that $A \rtimes_\alpha \mathbb{Z} = \{a, u : a \in A, u^*u = uu^* = 1, u^*au = \sigma(a)\} = \{\sum_n a_n u^n\}$. Suppose that $[P] \in K_0(A)$, then $[P] - [\alpha(P)] = [P] - [u^*Pu] = 0K_0(A \rtimes \mathbb{Z})$ (unitarily equivalent). $\square$

Proposition 1.37. Applying the above to the rotation algebra gives

$$\begin{aligned} A_\theta &= C^*\{u, v : uv = e^{2\pi i\theta} vu\} \\ &= C^*\{u, v : v^*uv = e^{2\pi i\theta} u\} \\ &= C^*\{C^*(u), v : v^*uv = e^{2\pi i\theta} u\} \\ &= C^*\{C^*(\mathrm{sp}(u)), v : v^*uv = e^{2\pi i\theta} u\} \\ &= C^*\{C^*(\mathbb{T}), v : v^*uv = e^{2\pi i\theta} u\} \\ &= C^*\{A, v : v^*av = \sigma(a)\} \text{ (translate the function } \theta) \\ &= C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}. \end{aligned}$$

Proposition 1.38. Let $A = C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, then $K_0(A) = \mathbb{Z}$. Let $u \in M_n(C(\mathbb{T})) = C(\mathbb{T} \to M_n)$. Hence $K_1(A) = \mathbb{Z}$. $[\alpha]_0 = 1$ and $[\alpha]_1 = 1$. It follows that

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \to K_0(A \rtimes \mathbb{Z}) \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to K_1(A \rtimes \mathbb{Z}) \to \mathbb{Z}.$$

Plus, we have

$$\begin{aligned} 0 \to \mathbb{Z} \to K_0(A \rtimes \mathbb{Z}) \to \mathbb{Z} \to 0, \\ 0 \to \mathbb{Z} \to K_1(A \rtimes \mathbb{Z}) \to \mathbb{Z} \to 0, \\ K_0(A \rtimes \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \\ K_1(A \rtimes \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

Proposition 1.39. Consider $0 \to K \to E \to A \to 0$, where K is a compact operator or $\ell^2(\mathbb{Z})$. Assume that $K \lhd E$ is essential, i.e., $a \in E$, a K $= \{0\}$ (or $ka = \{0\}$) $\implies a = 0 \implies \forall a \in E, L_a : K \ni b \to ab \in K, R_a : K \ni b \to ba \in K$. If $(L_a, R_a) = 0$, then $a = 0$.

Proposition 1.40. $E \curvearrowright B(H) \to B(H)/K \curvearrowleft A \overset{\pi}{\leftarrow} E$. $\eta_1, \eta_2 : A \curvearrowright B(H)/K$ then

$$B(H)/K \otimes M_2(\mathbb{C}) = \begin{bmatrix} B(H)/K & B(H)/K \\ B(H)/K & B(H)/K \end{bmatrix} = B(H \oplus H)/K \cong B(H)/K.$$

Proposition 1.41. $(\eta_1 + \eta_2) = \begin{bmatrix} \eta_1 & \\ & \eta_2 \end{bmatrix} \to \begin{bmatrix} B(H)/K & \\ & B(H)/K \end{bmatrix}$. Then $\eta_1 \sim \eta_2 : H \exists$unitary $u \in B(H)/K$ such that $\eta_1 = u^*\eta_2 u$.

Definition 1.36. $\mathrm{Ext}(A, K) := \{\eta : A \to B(H)/K\}/\sim$.

Definition 1.37. $\eta \in \mathrm{Ext}(A, K)$, $0 \to K \to E \to A \to 0 \implies K_0(K) \to K_0(E) \to K_0(A) \to K_1(A) \to K_1(E) \to K_1(A) \overset{\mathrm{Ind}}{\to} K_0(K) \implies \mathbb{Z} \to K_0(E) \to K_0(A) \to 0 \to K_1(E) \to K_1(A) \to \mathbb{Z}$, where Ind : $\mathrm{Ext}(A, K) \to \mathrm{Hom}(K_1(A) \to \mathbb{Z})$. If $\mathrm{Ind}(\eta) = 0$, then we have $0 \to \mathbb{Z} \to K_0(E) \to K_0(A) \to 0$ and $0 \to 0 \to K_1(E) \to K_1(A) \to 0$. In general, we have $0 \to \mathrm{Ext}_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \to \mathrm{Ext}(A, K) \to \mathrm{Hom}(K_1(A) \to \mathbb{Z}) \to 0$. It is true for $C(X)$ and $C(X) \otimes M_n$. If $0 \to I \to J \to K \to 0$, if any two of I, J, K satisfy UCT, the third one must satisfy UCT.

Question: Amenability implies UCT. (see more at Amenable group)

Proposition 1.42. $\forall \mathcal{F} \subset A, \forall \varepsilon > 0, \exists \phi : A \to M_n, \psi : M_n \to A$ such that $\|a - \psi\varphi(a)\| \leq \varepsilon, \forall a \in \mathcal{F}$.

Proposition 1.43. Γ is a discrete group, if $C^*(\Gamma)$ is amenable, then Γ is amenable.

Proposition 1.44. Any commutative von Neumann algebra $(L^\infty(X) \rtimes \Gamma)$ is in $L^\infty(X)$ and any commutative C^*-algebra $(C(X) \rtimes_{\mathrm{red}} \Gamma)$ is like $C(X)$. (Further reading: Tao's blog: von Neumann algebra, paper 1, Jones' paper, Wikipedia: Von Neumann algebra)

Proposition 1.45. $(X, \Gamma) \cong (X', \Gamma') \iff L^\infty(X) \rtimes \Gamma \cong L^\infty(X') \rtimes \Gamma'$ (orbit equivalence). (Further reading: Farah's paper: Turbulence, orbit equivalence, ..., Gaboriau: Orbit Equivalence and Measured Group Theory)

Proposition 1.46. Suppose that $\sigma : S^3 \to S^3$ and $\sigma' : S^5 \to S^5$, then it is minimal and uniquely ergodic. And we have $C(S^3) \rtimes_\sigma \mathbb{Z} \cong C(S^5) \rtimes_\sigma \mathbb{Z} \iff \mathrm{Ell}() \cong \mathrm{Ell}()$.(Further reading: Ergodicity, Dynamical System, Morphism, Homeomorphism, Homomorphism, Isomorphism, Monomorphism, Epimorphism, Bimorphism, Automorphism, Endormorphism, Meromorphic function, Holomorphic funciton, Homotopy, Isometry)

Definition 1.38. If $C(X) \rtimes Z$ is classifiable, do we have $\mathrm{mdim}(X, \sigma) = 0$ (min dimension)?

Definition 1.39. $S : A \to SA := C_0(\mathbb{R}) \otimes A = \} f : \mathbb{R} \to A, \lim_{t\to\infty} f(t) = 0\}$ is a suspension. Then

$$S\mathbb{C} = C_0(\mathbb{R}),$$

whose spectrum is $\mathbb{R}$. And we have $S(C_0(X)) = C_0(\mathbb{R}) \otimes C_0(X) = C_0(\mathbb{R} \times X)$.a For the short exact sequence,

$$0 \to C_(((0,1)) \to C_0((0,1]) \to \mathbb{C} \to 0,$$

we have

$$K_0(S\mathbb{C}) \to K_0(C_0(0,1]) \to K_0(\mathbb{C}) \to K_1(S\mathbb{C}) \to K_1(C_0(0,1]) \to K_1(\mathbb{C}) \to K_0(S\mathbb{C}).$$

Then it follows that

$$K_0(S\mathbb{C}) \cong K_1(\mathbb{C}), K_1(S\mathbb{C}) \cong K_0(\mathbb{C}).$$

Therefore,

$$0 \to C_0(\mathbb{R}) \otimes A \to C_0((0,1]) \otimes A \to A \to 0.$$

Hence, we have

$$K_0(SA) \to K_0(C_0(0,1]) \to K_0(A) \to K_1(SA) \to K_1(C_0(0,1]) \to K_1(A) \to K_0(SA).$$

Proposition 1.47.

$$K_*(C(\mathbb{T})) = K_*(C(S')) = K_*(C(\mathbb{R} \cup \infty)) = K_*(\widetilde{C_0(\mathbb{R})}) = K_*(\widetilde{S\mathbb{C}}).$$

Then

$$K_0(\widetilde{S\mathbb{C}}) = K_0(S\mathbb{C}) \oplus \mathbb{Z} \cong K_1(\mathbb{C}) \oplus \mathbb{Z} = \mathbb{Z}.$$

And

$$K_1(\widetilde{S\mathbb{C}}) = K_1(S\mathbb{C}) \cong K_0(\mathbb{C}) = \mathbb{Z}.$$

Further,

$$K_*(C(S^2)) = K_*(\widetilde{C_0(\mathbb{R}^2)}) = K_*(\widetilde{C_0(\mathbb{R}) \otimes C_0(R)}) = K_*(\widetilde{SC_0(\mathbb{R})}) = K_*(\widetilde{SS\mathbb{C}}).$$

It follows that

$$K_0(\widetilde{SS\mathbb{C}}) \cong K_0(SS\mathbb{C}) \oplus \mathbb{Z} \cong K_0(\mathbb{C}) \oplus \mathbb{Z} \cong \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z},$$

and

$$K_1(\widetilde{SS\mathbb{C}}) \cong K_1(SS\mathbb{C}) \cong K_1(\mathbb{C}) = 0.$$

(Further reading: Tangent bundle)

Proposition 1.48.

$$\begin{cases} \theta : K_1(A) \to K_0(SA), \\ \rho : K_0(A) \to K_1(SA). \end{cases}$$

Then $u \in M_n(A)$, we have

$$\begin{bmatrix} u & \\ & u^* \end{bmatrix} \overset{U_t}{\sim} 1_{2n}$$

Therefore, we can find that

$$\theta : \left[u_t^* \begin{bmatrix} 1_n & \\ & 0 \end{bmatrix}_0 u_t \right]_0 - \left[\begin{bmatrix} 1_n & \\ & 0 \end{bmatrix} \right]_0$$

and

$$\rho : P \in M_n(A) \to t \mapsto e^{2\pi it}P + (1_n - P).$$

Suppose that θ is isomorphism, then $K_0(A) \cong K_1(SA)$. Then we have

$$K_1(A) = K_0(SA), K_2(A) = K_0(SSA), \ldots, K_n(A) := K_0(S \cdots SA).$$

$$0 \to I \to J \to K \to 0 \implies \cdots K_2(I) \to K_2(J) \to K_2(K) \overset{\text{Ind}}{\to} K_1(I) \to K_1(J) \to K_1(K) \overset{\text{Ind}}{\to} K_0(I) \to K_0(J) \to K_0(K),$$

where $K_0(K) \cong K_2(K)$.

2 Cuntz algebra

Definition 2.1. A C^*-algebra $\mathcal{A}$ looks stably like a *Cuntz-Krieger algebra* if $\mathcal{A}$ is seperable, nuclear and $\mathcal{O}_\infty$-absorbing, $\mathcal{A}$ has real rank zero and finitely many ideals, and every simple subquotient of $\mathcal{A}$ satisfies the universal coefficient theorem and has a finitely generated, free K_1-group of the same rank as its K_0-group. A C^*-algebra $\mathcal{A}$ looks like a Cuntz-Krieger algebra if it is unital and looks stably like a Cuntz-Krieger algebra. That is,

$$\mathcal{O}_n = C^*\{v_1, v_2, \ldots, v_n : v_i^* v_i = 1, v_1 v_1^* + v_2 v_1^* + \cdots + v_n v_n^* = 1\},$$

and

$$\mathcal{O}_\infty = C^*\{v_1, v_2, \ldots, v_n, \ldots : v_i^* v_i = 1, v_i v_i^* \perp v_j v_j^* \forall i \neq j\}.$$

Example 2.1. Let $\mathcal{O}_2 = \{v_1, v_2 : v_1^* v_1 = v_2^* v_2 = 1, v_1 v^* + v_2 v_2^* = 1\}$. Then consider $\mathcal{O}_2 \to \mathcal{O}_2 : a \mapsto v_1 a v_1^*, a \mapsto v_2 a v_2^*$. Then

$$M_{2^\infty} \cong C^*\{\sum (v_{n_1} v_{n_2} \cdots v_{n_k})(v_{m_1} v_{m_2} \cdots v_{m_k})^*\}.$$

Then $\mathcal{O}_2$ can be generated by M_{2^∞}.

$$\mathcal{O}_2 = C^*\{\sum (v_{n_1} v_{n_2} \cdots v_{n_l})(v_{m_1} v_{m_2} \cdots v_{m_t})^*\}.$$

Follows that $\exists E : \mathcal{O}_2 \to M_{2^\infty}$. Hence, $E(abc) = aE(b)c$, where $a, b \in M_{2^\infty}$. E is called unital completely positive map. Furthermore, $\forall z \in \mathbb{T}, \sigma_z : v_1 \to z v_1, v_2 \to z v_2$, where $\sigma_z \in \operatorname{Aut}(\mathcal{O}_2)$, $\mathbb{T} \curvearrowright \operatorname{Aut}(\mathcal{O}_2)$. Then $E(a) = \int_{\mathbb{T}} \sigma_z(a) dz \in \mathcal{O}_2$. We can conclude that $\mathcal{O}_2$ is also simple. $(M_{2^\infty} \otimes \mathcal{K}) \rtimes_\sigma \mathbb{Z} \cong \mathcal{O}_2 \otimes \mathcal{K}$.

(Futher reading: Extensions of Cuntz-Krieger Algebras, completely positive map, tensor product, Ergodic theory, partial isometry, Grothendieck group)

Proposition 2.1. The following is true,

$$K_*(M_{2^\infty} \otimes \mathcal{K}) \cong K_*(M_{2^\infty}).$$

And we have

$$M_2 \to M_{2^2} \to \cdots \to M_{2^\infty}, a \mapsto \begin{bmatrix} a & \\ & a \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} a & \\ & a \end{bmatrix} & \\ & \begin{bmatrix} a & \\ & a \end{bmatrix} \end{bmatrix} \mapsto \cdots.$$

Therefore, we have

$$K_*(M_2) \to K_*(M_{2^2}) \to K_*(M_{2^3}) \to \cdots \to K_*(M_{2^\infty}).$$

Hence, when $* = 0$, we have

$$\mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \overset{\times 2}{\to} \cdots \overset{\times 2}{\to} \mathbb{Z}[\frac{1}{2}].$$

It follows that

$$G_1 \overset{\phi_1}{\to} G_2 \overset{\phi_2}{\to} \cdots \overset{\phi_n}{\to} G_n \overset{\phi_{n+1}}{\to} \cdots \to \lim(G_n).$$

Then,

$$K_0(A \otimes B) \cong [K_0(A) \otimes K_0(B)] \oplus [K_1(A) \otimes K_1(B)] (\text{torsion free}).$$

Definition 2.2. $[\sigma]_0 : \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}] = \{\frac{m}{2^n}, m, n \in \mathbb{Z}\}$. P-V six-term exact sequence,

$$K_0(M_{2^\infty}) \overset{1-[\sigma]_0}{\to} K_0(M_{2^\infty}) \to K_1(M_{2^\infty}) \overset{1-[sigma]_0}{\to} K_1(M_{2^\infty}) \to K_1(\mathcal{O}_2) \to K_0(M_{2^\infty}).$$

Then

$$K_0(M_{2^\infty}) \overset{1-[\sigma]_0}{\to} K_0(M_{2^\infty}) \to K_1(M_{2^\infty}) \overset{1-[sigma]_0}{\to} 0 \to 0 \to K_0(M_{2^\infty}).$$

(Further reading: Approximately finite-dimensional C*-algebra, AF-algebra, Supernatural number, Torsion group)

Proposition 2.2. For a graph with only one vertex and an edge $\{p, v : vv^* = p, v^*v = p\} = \{v : v^*v = vv^* = 1\} = C(\mathbb{T})$.

Definition 2.3. $\pi : A \to B(H)$ is irreducible if the only (closed) invariant subspace of H are $\{0\}$ and $\{H\}$.

Theorem 2.1 (Schur)**.** π is irreducible if and only if

$$[\pi(A)]' = \mathbb{C}1,$$

where $\pi(A) = \{C \in B(H), [C, \pi(a)] = 0, \forall a \in A\}$.

Proposition 2.3. Let $\pi : A \to B(H)$ be irreducible, consider

$$\pi(v) \in [\pi(A)]',$$

then

$$\pi(v) = \lambda 1 = e^{i\theta}1.$$

In other words, $\pi : \xi \to e^{i\theta}$

Proposition 2.4. Now let's look at a points with two vertices and two edges,

$$C^*\{P_1, P_2 : P_1 + P_2 = 1, v_1v_1^* = P_2, v_1^*v_1 = P_1, v_2v_2^* = P_1, v_2^*v_2 = P_2\}$$

$$\tilde{P}_1 = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \tilde{P}_1 = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \tilde{V}_1 = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \tilde{V}_1 = \begin{bmatrix} & e^{i\theta} \\ 0 & \end{bmatrix}$$

1. Find central element c of A.
2. Let $\pi : A \to B(H)$ be irreducible, consider $\pi(c) = \mathbb{C}1$.
3.

(Further reading: Group action, Group action (UConn), Group actions (Brilliant), Irreducible representation, Partial isometries, Partial isometry (Wikipedia))

Proposition 2.5. Assume that A is a unital C^*-algebra. Let $V_0(A)$ be the semigroup of the isomorphism classes of f, g projection module, which is the direct summand of $A \oplus \cdots \oplus A = \gamma(A^n)$. Then the $K_0(A)$ is an enveloping group of $V_0(A)$, $[p] - [q]/\sim$.

Definition 2.4. Consider $B(H)$,

- $\|\cdot\|$ uniform convergence topolgy, i.e., $a_n \to a \iff \forall \varepsilon \exists N$ such that $\|a_n(\xi) - a(\xi)\| < \varepsilon, \forall \xi \in B(H), \|\xi\| = 1, \forall n > N$.
- Pointwise convergence: $a_n \to a$ pointwisely, i.e., $\forall \xi, a_n(\xi) \to a(\xi)$ in $\|\cdot\|$, that is, $\langle a_n(\xi) - a(\xi), a_n(\xi) - a(\xi)\rangle \to 0$. Equivalently, $\forall \xi \langle a_n(\xi), \eta\rangle \to \langle a(\xi), \eta\rangle, \forall \eta$.

Definition 2.5. $\mu \subset B(H)$ is a von Neumann algebra if μ is closed under either strong operator topology or the weak operator topology.

Example 2.2. $T \in B(H), T = T^* = T^*T$, and the spectral projection

$$T = \int_{sp(T)} \lambda \, dE(\lambda).$$

Then

$$C^*(T, 1) = \overline{\{\sum c_n T^n\}} = C(\text{sp}(T)).$$

Thus,

$$vN(T) = L^\infty(\text{sp}(T)).$$

Theorem 2.2. $1 \in \mu \subset B(H)$, then $\overline{\mu}$ is von Neumann algebra. Then

$$\overline{\mu} = \mu'' = (\mu')'.$$

Proposition 2.6. Let Γ be a discrete group, $\ell^2(\Gamma) = \{f : T \to \mathbb{C}, \sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty\}$. Let $\pi_r : \Gamma \to B(\ell^2(\Gamma))$ be the left shifting, i.e., $\gamma \mapsto (\gamma' \mapsto \xi(r')) \mapsto (r' \mapsto \xi(\gamma'\gamma))$.

$$R(\Gamma) := \pi_r(\Gamma)'', L(\Gamma) := \pi_l(\Gamma)'' \implies R(\Gamma)' = L(\Gamma)'.$$

Further, $L(\Gamma)$ has a faithful trace, $a \mapsto \langle a\xi_e, \xi_e \rangle$.

Theorem 2.3. If Γ is an ICC group, then $L(\Gamma)$ is a II_1-factor.

Definition 2.6. μ is a factor if $\mu \cap \mu' = \mathbb{C}1$.

Proposition 2.7. If μ is a factor, then

1. $\mu = B(H)$.
2. μ has a faithful trace, it is II_1 if the trace is bounded, II_∞ if the trace is unboudned.

Example 2.3. $F_n, n \geq 2$, and $S_\infty = \bigcup_n S_n$.

Theorem 2.4. $L(S_\infty)L(F_n), \forall n > N$.

Open problem: $L(F_n) \cong L(F_m), m \neq n$?
(Further reading: Faithful representation, Abelian von Neumann algebra, Infinite conjugacy class property, A gentle introduction to von neumann algebras for model theorists)