MATH 5200 - Real Variables Lecture Notes 1

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1 Outline

- Lebesgue measure on $\mathbb{R}$ ($\mathbb{R}^n$) and Lebesgue integral.
- Abstract measure and Lebesgue integral.
- Convergence theorems: $\displaystyle\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$.
- Differentiation and density theorem.
- Recommendation for point set topology: *Basic Topology* by Armstrong.

2 Lebesgue measure (concrete measure) on $\mathbb{R}$ ($\mathbb{R}^n$)

- Question: what is the measure/magnitude of the following objects?
  - $m([0,1]) = 1$.
  - $m([0,1)) = 1$.
  - $m(\{1\}) = 0$.
  - m(square with side equals 1) $= 1$.
  - m(rectangle whose width and length are 1 and 2 respectively) $= 1$.
  - m(circle with radius 1) $= 1$.
  - $m(\mathbb{Q}) = 0$?
  - $m([0,1]\backslash\mathbb{Q}) = 1$?
  - m(Cantor set) $= 0$?
  - m(higher dimensional Cantor set) $= 0$?

2.1 Elementary sets

- **Definition**:
  - **Interval**: $I = [a, b]$, (a, b), $(a, b]$, $[a, b)$, which allows $a = b$, that is to say, single point is an interval, so is empty set. The measure (length) of I: $|I| = b - a$.
  - **Box**: $B = I_1 \times I_2 \times I_3 \times \cdots \times I_n \subseteq \mathbb{R}^n$ (the product of finitely many intervals). The measure (volume) of B: $|B| = |I_1| \times |I_2| \times |I_3| \times \cdots \times |I_n|$.
  - **Elementary set**: an elementary set is a finite union of intervals ($\mathbb{R}$) or boxes ($\mathbb{R}$).
- **Lemma**: Let $E = \bigcup_{i=1}^n I_n$ be an elementary set.
  1) E can be written as disjoint union of finitely many intervals ($\mathbb{R}$) or boxes ($\mathbb{R}^n$).
  2) If
  $$\begin{cases} E = I_1 \cup I_2 \cup I_3 \cup \cdots \cup I_s, \\ E = J_1 \cup J_2 \cup J_3 \cup \cdots \cup J_t, \end{cases}$$
  then
  $$m(E) = |E| = |I_1| \times |I_2| \times |I_3| \times \cdots \times |I_s| = |J_1| \times |J_2| \times |J_3| \times \cdots \times |J_t|$$

2.2 Jordan measure

- **Definition**: Let $E \subseteq \mathbb{R}^n$ be a bounded set.
  - The Jordan inner measure $m_{*,(J)}(E)$ of E is defined as
  $$m_{*,(J)}(E) := \sup_{A \subset E, A \text{ elementary}} m(A).$$
  - The Jordan outer measure $m^{*,(J)}(E)$ of E is defined as
  $$m^{*,(J)}(E) := \inf_{B \supset E, B \text{ elementary}} m(B).$$
  - If $m_{*,(J)}(E) = m^{*,(J)}(E)$, then we say that E is *Jordan measurable*, and call $m(E) := m_{*,(J)}(E) = m^{*,(J)}(E)$ the *Jordan measure* of E.
- **Examples**
  1) Elementary sets are Jordan measurable.
  2) E is a sector of circle. To show E is Jordan measurable, need to show $\forall \epsilon > 0, \exists A \subseteq E \subseteq B, m(B) - m(A) < \epsilon$, where A and B are both elementary sets. The methods is something similar to Riemann sum, $m(B) - m(A) = \frac{1}{n}[f(\frac{0}{n}) - f(\frac{1}{n})] + \frac{1}{n}[f(\frac{1}{n}) - f(\frac{2}{n})] + \cdots + \frac{1}{n}[f(\frac{n-1}{n}) - f(\frac{n}{n})] = \frac{1}{n}[f(0) - f(1)] \to 0$ as $n \to \infty$ (telescoping sum).
- **Lemma**: E is Jordan measurable if and only if $\forall \epsilon > 0$, there are elementary sets $A \subseteq E \subseteq B$ such that $m(B \backslash A) = m(B) - m(A) < \epsilon$, if and only if $\forall \epsilon > 0$, there is an elementary set A such that $m^{*,(J)}(E \triangle A) < \epsilon$, where $E \triangle A = (E \backslash A) \cup (A \backslash E)$ (symmetric difference).
- **Examples**:
  1) $E = \mathbb{Q} \cap [0, 1]$. $m_{*,(J)}(E) := \sup_{A \subset E, A \text{ elementary}} m(A)$, if $A \subseteq \mathbb{Q}$, A elementary, then A = finitely many point $\Rightarrow$ $m_{*,(J)}(E) = 0$. While $m^{*,(J)}(E) := \inf_{B \supset E, B \text{ elementary}} m(B)$, E is dense in $[0, 1]$, then $m^{*,(J)}(E) = m(A) = 1$.
  2) $E = [0, 1] \backslash \mathbb{Q}$. $m_{*,(J)}(E) = 0$ and $m^{*,(J)}(E) = 1$.
  3) $Q \cap [0, 1] = r_1, r_2, \ldots, r_n, \ldots$, $E = (r_1 - \epsilon, r_1 + \epsilon) \cup (r_2 - \frac{\epsilon}{2}, r_2 + \frac{\epsilon}{2}) \cup (r_3 - \frac{\epsilon}{4}, r_3 + \frac{\epsilon}{4}) \cup \cdots \cup (r_n - \frac{\epsilon}{2^{n-1}}, r_n + \frac{\epsilon}{2^{n-1}}) \cup \cdots$. $m_{*,(J)}(E) \leq 2\epsilon + \epsilon + \frac{\epsilon}{2} + \cdots = \lim_{n \to \infty} \epsilon(\frac{2 - (1/2)^n}{1 - 1/2}) = 4\epsilon$. While $m^{*,(J)}(E) = 1$ because E is dense.
- **Properties of elementary measure**: Let $E, F \subset \mathbb{R}^n$ be Jordan measurable sets.
  1) (Boolean closure) $E \cup F$, $E \cap F$, $E \backslash F$, and $E \triangle F$ are Jordan measurable.
  2) (Non-negativity) $m(E) \geq 0$.
  3) (Finite additivity) If E, F are disjoint, then $m(E \cup F) = m(E) + m(F)$.
  4) (Monotonicity) If $E \subset F$, then $m(E) \leq m(F)$.
  5) (Finite subadditivity) $m(E \cup F) \leq m(E) + m(F)$.
  6) (Translation invariance) For any $x \in \mathbb{R}^n$, $E + x$ is Jordan measurable, and $m(E + x) = m(E)$.
- Recall that $E = \mathbb{Q} \cap [0, 1]$ is not Jordan measurable, because $m_{*,(J)}(E) = 0$, $m^{*,(J)}(E) = 1$ (because we are restricted to finite covers).

2.3 Lebesgue measure

- **Definition**: Let $E \subseteq \mathbb{R}^n$. Then Lebesgue outer measure of E is

$$m^*(E) := \inf_{\bigcup_{n=1}^{\infty} B_n \supset E; B_1, B_2, \ldots \text{ boxes}} \sum_{n=1}^{\infty} |B_n|,$$

or

$$m^*(E) := \inf_{\bigcup_{n=1}^{\infty} I_n \supset E; I_1, I_2, \ldots \text{ intervals}} \sum_{n=1}^{\infty} |I_n| = r,$$

$$m^*(E) := \inf_{\bigcup_{n=1}^{\infty} J_n \supset E; J_1, J_2, \ldots \text{ open intervals}} \sum_{n=1}^{\infty} |J_n| = r'.$$

- **Lebesgue measurability**: A set $E \subset \mathbb{R}^n$ is said to be *Lebesgue measurable* if, for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ containing E such that $m^*(U \backslash E) \leq \epsilon$. If E is Lebesgue measurable, we refer to $m(E) = m^*(E)$ as the *Lebesgue measure* of E (note that this quantity may be equal to $+\infty$). We also write $m(E)$ as $m^*(E)$ when we wish to emphasize the dimension n.

- **Example**: $\mathbb{Q} = \{r_1, r_2, \ldots, r_n, \ldots\}$, we have $m^*(\mathbb{Q}) = 0$. $\forall \epsilon > 0$, consider $\bigcup_{n=1}^{\infty} (r_n - \frac{\epsilon}{2^{n+1}}, r_n + \frac{\epsilon}{2^{n+1}}) \supseteq \mathbb{Q}$, which implies $\sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$, hence $m^*(\mathbb{Q}) = 0$. Note: it also works for all the countable sets.

- Now let's show $r = r'$. First, $r \leq r'$, since we have more candidates in the first case, hence, the infimum is smaller, i.e. $r \leq r'$. Next, we need show $r \geq r'$, which is equivalent to show $r + \epsilon \geq r', \forall \epsilon > 0$). Fix $\epsilon > 0$, pick $\bigcup_{n=1}^{\infty} I_n$, where I_n are intervals which may not be open, such that $r \leq \sum_{n=1}^{\infty} |I_n| \leq r + \frac{\epsilon}{2}$. Consider $\bigcup_{n=1}^{\infty} I_n$, for each I_n, choose $\tilde{I}_n \supseteq I_n$ such that $\tilde{I}_n$ is open and $|\tilde{I}_n| - |I_n| < \frac{\epsilon}{2^{n+1}}$. Consider $\bigcup_{n=1}^{\infty} \tilde{I}_n \supseteq E$, hence, we have $r' \leq \sum_{n=1}^{\infty} |\tilde{I}_n| \leq \sum_{n=1}^{\infty} |I_n| + \frac{\epsilon}{2} \leq r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon$.

- **Properties of Lebesgue outer measure** (the outer measure axioms):

  1) (Empty set) $m^*(\emptyset) = 0$.
  2) (Monotonicity) If $E \subset F \subset \mathbb{R}^n$, then $m^*(E) \leq m^*(F)$.
  3) (Countable subadditivity) If $E_1, E_2, \ldots \subset \mathbb{R}^n$ is a countable sequence of sets, then $m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

- **Remark**:

  – $m^*(E \cup F) = m^*(E) + m^*(F)$? If $E \cap F = \emptyset$.
  – $m^*(E \cup F) = m^*(E) + m^*(F)$, if $\text{dist}(E, F) = \delta > 0$, where $\text{dist}(E, F) = \inf\{\text{dist}(x, y), x \in E, y \in F\}$.

- **Lemma**: If $\text{dist}(E, F) = \delta > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.
  Proof. By countable subadditivity, $m^*(E \cup F) \leq m^*(E) + m^*(F)$. Next, we need to show $m^*(E \cup F) \geq m^*(E) + m^*(F)$, which is equivalent to show $m^*(E \cup F) + \epsilon \geq m^*(E) + m^*(F), \forall \epsilon > 0$. For $E \cup F$, choose $\bigcup_{n=1}^{\infty} I_n$ such that $m^*(E \cup F) \leq \sum_{n=1}^{\infty} |I_n| \leq m^*(E \cup F) + \epsilon$. Refine $I_n, n = 1, 2, \ldots$ such that $\text{diam}(I_k) < \delta$. Then take $C_E = \{n | I_n \cap E \neq \emptyset\}$ and $C_F = \{n | I_n \cap F \neq \emptyset\}$, note: $C_E \cap C_F = \emptyset$.

$$\bigcup_{n \in C_E} I_n \supseteq E \text{ and } \bigcup_{n \in C_F} I_n \supseteq F \Rightarrow m^*(E) + m^*(F) \leq \sum_{n \in C_E} |I_n| + \sum_{n \in C_F} |I_n| \leq \sum_{n=1}^{\infty} \leq m^*(E \cup F) + \epsilon.$$

.

- **Lemma 1.2.6**: (Outer measure of elementary sets) Let E be an elementary set. Then the Lebesgue outer measure $m^*(E)$ of E is equal to the elementary measure $m(E)$ of E: $m^*(E) = m(E)$.

  Proof. We already know that $m^*(E) \le m^{*,(J)}(E) = m(E)$, so it suffices to show that $m(E) \le m^*(E)$. It's enough to show that $m(E) \le m^*(E) + \epsilon, \forall \epsilon > 0$. Assume that E is compact (closed and bounded) subset of $\mathbb{R}^n$. For the given $\epsilon > 0$, pick $\bigcup_{n=1}^{\infty} I_n \supseteq E$ such that $\sum_{n=1}^{\infty} |I_n| \le m^*(E) + \epsilon$, since E is compact, $\exists I_{n_1}, I_{n_2}, \dots, I_{n_K}$ such that $\bigcup_{k=1}^{K} I_{n_k} \supseteq E$. Then $m(E) \le m(\bigcup_{k=1}^{K} I_{n_k}) \le \sum_{k=1}^{K} |I_{n_k}| \le \sum_{n=1}^{\infty} |I_n| \le m^*(E) + \epsilon$.

- **Lemma**: If $U \subseteq \mathbb{R}$ is open, then $U = \bigcup_{i=1}^{\infty} (a_i, b_i)$.

- **Lemma**: Let $U \subseteq \mathbb{R}^n$ be an open set. Then $U = \bigcup_{n=1}^{\infty} I_n$, where I_n are closed boxes, and $I_m^{\text{int}} \cap I_n^{\text{int}} = \emptyset, m \ne n$.

- **Example**: an open set is can be covered by countably many boxes.

  Step 1: use a grid to cover the set, let the size of the boxes be 1, then cover those area which is covered by the whole box.

  Step 2: refine the grid by setting the size of the boxes to $\frac{1}{2}$, then pick those boxes which are covered by the whole boxes.

  Step 3: repeat step 2 until all the points can be covered by the boxes (countably many steps).

  Since it is an open set, according to the definition of open set, every point inside of set, the small neighborhood of the point must be inside of the set. Hence, we can hit every point in countably many steps.

- **Lemma**: If U is open, $m^*(U) = m_{*,(J)}(U)$.

- **Lemma**: (outer regularity) Let E be an arbitrary set. Then $m^*(E) = \inf_{U \supseteq E, U \text{ open}} m^*(U)$.

2.4 Lebesgue measurability

- Recall: $E \supseteq \mathbb{R}^n$ is measurable if $\forall \epsilon > 0, \exists U \supseteq E$, where U is open such that $m^*(U \backslash E) \le \epsilon$.

- Denote $\mathcal{M}(\mathbb{R}^n) = \{E \subseteq \mathbb{R}^n, E \text{ measurable}\}$.

- **Lemma 1.2.13**: (Existence of Lebesgue measurable sets)

  (0) Empty set $\emptyset \in \mathcal{M}(\mathbb{R}^n)$.
  (1) If U is open, then $U \in \mathcal{M}(\mathbb{R}^n)$.
  (2) If D is closed, then $D \in \mathcal{M}(\mathbb{R}^n)$.
  (3) If $E \in \mathcal{M}(\mathbb{R}^n)$, then $E^{\complement} \in \mathcal{M}(\mathbb{R}^n)$.
  (4) If $E_1, E_2, \dots, E_n, \dots \in \mathcal{M}(\mathbb{R}^n)$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}(\mathbb{R}^n)$.
  (5) If $E_1, E_2, \dots, E_n, \dots \in \mathcal{M}(\mathbb{R}^n)$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}(\mathbb{R}^n)$.

  Proof. (0) is trivial; (1) is trivial; (2) is trivial if we assume (3); (5) is trivial if we assume (3) and (4):

$$\bigcap_{n=1}^{\infty} E_n = \left[\left(\bigcap_{n=1}^{\infty} E_n \right)^{\complement} \right]^{\complement} = \left(\bigcup_{n=1}^{\infty} E_n^{\complement} \right)^{\complement}.$$

  (4): Let $E_1, E_2, \dots, E_n, \dots$ be measurable sets. Pick open sets $U_1 \ge E_1, U_2 \ge E_2, \dots, U_n \ge E_n, \dots$ such that $m^*(U_1 \backslash E_1) \le \frac{\epsilon}{2}, \dots, m^*(U_n \backslash E_n) \le \frac{\epsilon}{2^n}, \dots$. Consider $U = \bigcup_{n=1}^{\infty} U_n$, where $U \supseteq \bigcup_{n=1}^{\infty} E_n$ is open, then we have

$$U \backslash \bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} U_n \right) \backslash \left(\bigcup_{n=1}^{\infty} E_n \right) \subseteq \bigcup_{n=1}^{\infty} (U_n \backslash E_n).$$

Therefore,

$$m^*(U\backslash\bigcup_{n=1}^{\infty}E_n) \leq m^*(\bigcup_{n=1}^{\infty}U_n\backslash E_n) \leq \sum_{n=1}^{n} m^*(U_n\backslash E_n) \leq \sum_{n=1}^{\infty}\frac{\epsilon}{2^n} = \epsilon.$$

(2): To show closed sets are measurable.

Proof. Let E be closed and bounded (compact). Fix $\epsilon > 0$, pick $U \geq E$ such that $m^*(U) \leq m^*(E) + \epsilon$. Let's show that $m^*(U\backslash E) < \epsilon$. Note that $U\backslash E$ is open. Then $U\backslash E = \bigcup_{n=1}^{\infty} I_n$, where I_n is compact, $I_n^{\text{int}} \cap I_m^{\text{int}} = \emptyset$, $n \neq m$. For an arbitrary N, consider $\text{dist}(\bigcup_{n=1}^{N} I_n, E) = \delta > 0$.

$$m^*(\bigcup_{n=1}^{N}) + m^*(E) = m^*\left[\left(\bigcup_{n=1}^{N} I_n\right) \cup E\right] \leq m^*(U) leq m^*(E) + \epsilon.$$

Then we have $\sum_{n=1}^{N} |I_n| = m^*(\bigcup_{n=1}^{N} I_n) \leq \epsilon, \forall N \Rightarrow \sum_{n=1}^{\infty} |I_n| \leq \epsilon$. Note: closed and unbounded set can be $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n]$, $\mathbb{R}^d = \bigcup_{n=1}^{\infty} \overline{B}_n$. Let E be measurable, consider $E^{\complement}$, $\forall\epsilon > 0$, there is an open set $U \supseteq E$ such that $m^*(U\backslash E) < \epsilon$. Hence there is open set $V \supset U\backslash E$ such that $m^*(V) < \epsilon$. Consider $U^{\complement}$. This is a closed set, hence measurable. So $\exists W \supseteq U^{\complement}$ such that $m^*(W\backslash U^{\complement}) < \epsilon$. Consider $W \cup V$, which is open, we have

$$W \cup V \supseteq U^{\complement} \cup (U\backslash E) \supseteq E^{\complement}.$$

Let $A = (W \cup V)\backslash E^{\complement} \subseteq (W\backslash U^{\complement}) \cup V$, if $x \in A$, then $x \in W$ or $x \in V$ and $x \in E^{\complement}$ ($x \in E$). Then we have

$$m^*[(W \cup V)\backslash E^{\complement}] \leq m^*[(W\backslash U^{\complement}) \cup V] \leq m^*(W\backslash U^{\complement}) + m^*(V) \leq 2\epsilon$$

(3): Let $E \in \mathcal{M}(\mathbb{R}^n), \forall\epsilon > 0$. Pick $U \geq E$ such that $m^*(U\backslash E) < \epsilon$. Pick $V = \bigcup_{n=1}^{\infty} I_n \supseteq U\backslash E$ such that $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Need an open set $W \supseteq U\backslash E^{\complement}$, $m^*(W\backslash E^{\complement}) < \epsilon$. A natural candidate is $V \cup U^{\complement}$, $V \cup O$, where $O \supseteq U^{\complement}$, and O is open and $U^{\complement}$ is closed.

- **Completeness**: (1) If $m^*(E) = 0$, then E is measurable. (2) If $m^*(E) = 0$, then any subset of E is measurable.

  Proof. If $m^*(E) = 0$, then there is open set $U \supseteq E$ such that $m(U) \approx m^*(E) = 0$, $m^*(U\backslash E) \leq m^U \leq \epsilon$. So E is measurable.

- Let C be the standard Cantor set. By exercise we have $m(C) = 0$. Then any subset of C is measurable. $2^c \subseteq \mathcal{M}(\mathbb{R}) \subseteq 2^{\mathbb{R}}$, $|\mathcal{M}(\mathbb{R})| = |2^{\mathbb{R}} = 2^c$.

- **Conclusion**:
  - $\mathcal{M}(\mathbb{R}^d)$ is a σ-algebra and
  - If $m^*(E) = 0$, then $E \in \mathcal{M}(\mathbb{R}^d)$.

- **Theorem**: If $E_1, E_2, \ldots, E_n, \ldots$ measurable, and $E_i \cap E_j \neq \emptyset$, then $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

- Recall: if E, F, $\text{dist}(E, F) = \delta > 0$, we have $m^*(E \cup F) = m^*(E) + m^*(F)$.

- **Proposition**: The following are equivalent:
  1. E is measurable.
  2. $\forall\epsilon > 0$, $\exists U \supseteq E$ such that $m^*(U\backslash E) < \epsilon$.
  3. $\forall\epsilon > 0$, $\exists U$ open such that $m^*(U\Delta E) < \epsilon$.
  4. $\forall\epsilon > 0$, $\exists D \subseteq E$, where D is closed, such that $m^*(D\backslash E) < \epsilon$.
  5. $\forall\epsilon > 0$, $\exists D$ such that $m^*(D\Delta E) < \epsilon$.

  Notice: (2) $\Rightarrow$ (3) is trivial. Look at the case (3) $\Rightarrow$ (2), pick W open, $W \supseteq U\Delta E$ such that $m(W) < \epsilon$. Consider $U' := U \cup W \geq E$, we have $U'\backslash E \supset W$.

- Given E, consider $E^{\complement}$ (measurable). $\exists U \supseteq E^{\complement}$, $m^*(U \backslash E) < \epsilon$. Then $D := U^{\complement} \subseteq (E^{\complement})^{\complement}$.

  Proof. $m(\bigcup_{i=1}^{\infty} E_i) \leq m(E_1) + m(E_2) + \cdots + m(E_n) + \cdots$. Let's prove that $m(\bigcup_{i=1}^{\infty} E_i) + \epsilon \geq m(E_1) + m(E_2) + \cdots$. Assume that all E_i are closed and bounded (compact), then

$$m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i), \forall N \in \mathbb{Z}$$
$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{N} m(E_i), \forall N \in \mathbb{Z}$$
$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$$

- It is known that $E_1, E_2, \ldots, E_n, \ldots$, $\mathrm{dist}(E_i, E_j) = \delta_{ij} > 0$. Then $m^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$.

- Assume E_i are bounded, measurable. Then, pick

$$D_1 \subseteq E_1, D_1 \text{ closed and } m(E_1 \backslash D_1) < \frac{\epsilon}{2}$$
$$\vdots$$
$$D_n \subseteq E_n, D_n \text{ closed and } m(E_n \backslash D_n) < \frac{\epsilon}{2^n}$$

$m(\bigcup_{i=1}^{N} E_i) \geq m(\bigcup_{i=1}^{N} D_i) = m(D_1) + m(D_2) + \cdots + m(D_n) = m(E_1) + m(E_2) + \cdots + m(E_n)$. Then $m(\bigcup_{i=1}^{N} E_i) + \epsilon \geq m(E_1) + m(E_2) + \cdots + m(E_n)$

- Decompose each E_n into disjoint and boudned pieces.

- **Summary**

  - $\mathcal{M}(\mathbb{R}^d)$ is a σ-algebra, i.e.
    (1) $\emptyset \in \mathcal{M}(\mathbb{R}^d)$.
    (2) $E \in \mathcal{M}(\mathbb{R}^d) \Rightarrow E^{\complement} \in \mathcal{M}(\mathbb{R}^d)$.
    (3) $E_1, E_2, \ldots, E_n, \ldots \in \mathcal{M}(\mathbb{R}^d) \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}(\mathbb{R}^d)$.
  - $m : \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^+ \cup \{0\}$.
    (1) $m(\emptyset) = 0$.
    (2) $m(E) \geq m(F)$ if $E \supseteq F$.
    (3) $m(\bigcup_{i=1}^{\infty} E_i) \leq m(E_1) + \cdots + m(E_n) + \cdots$, where $E_i \in \mathcal{M}(\mathbb{R}^d)$.
    (4) $m(\bigcup_{i=1}^{\infty} E_i) = m(E_1) + \cdots + m(E_n) + \cdots$, if $E_i \cap E_j = \emptyset, i \neq j$.

- **Conclusion**: If $m(E) = 0$, and $F \subseteq E$, then $F \subseteq E$.

- **A nonmeasurable set**: Consider $\mathbb{T} = \mathbb{R} \backslash \mathbb{Z}$. Let $m(\mathbb{T}) = 1$, Let $Q = \mathbb{Q} \backslash \mathbb{Z} \subseteq \mathbb{T}$. Then $0 \to \mathbb{Q} \to \mathbb{T} \to \mathbb{T} \backslash \mathbb{Q} \to 0$. (Set theoretical lifting). Pick a representative in each class of $\overline{t + Q}$. Let E be the set of representative. Then

  (1) $\mathbb{T} = \bigcup \overline{t + Q} = \bigcup_{r_i \in \mathbb{Q}} (E + r_i)$.
  (2) $(E + r_i) \cap (E + r_j) = \emptyset$, if $r_i \neq r_j$. Why? If $s \in (E + r_i) \cap (E + r_j) \Rightarrow s = e + r_i = e' + r_j$. That is $e - e' = r_j - r_i \in \mathbb{Q}$.

(3) $m(E + r_i) = m(E)$ if E is measurable. $1 = m(\mathbb{T}) = m(\bigcup_{r_i \in \mathbb{Q}} (E + r_i)) = m(E + r_1) + m(E + r_2) + \cdots = m(E) + m(E) + m(E) + \cdots = m(E) \cdot \infty$. This case is unrealistic. Hence $\mathbb{T}$ is not measurable.

- **Theorem**: If $E_1, E_2, \ldots, E_n, \ldots$ measurable, disjoint, then $m(E_1 \cup \cdots \cup E_n \cup \cdots) = \sum_{n=1}^{\infty} m(E_n)$.

- **Corollary**: If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \subseteq$ measurable sets then $m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n) = \sup_{n=1,2,\ldots} (m(E_n))$.

  Proof. Assume $m(E_n) < \infty$ for all $n \in \mathbb{N}$. We have

$$\bigcup_{n=1}^{\infty} E_n = E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash E_2) \cup \cdots \cup (E_n \backslash E_{n-1}) \cup \cdots .$$

  Then

$$m(\bigcup_{n=1}^{\infty} E_n) = m(E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash E_2) \cup \cdots \cup (E_n \backslash E_{n-1}) \cup \cdots) = m(E_1) + m(E_2 \backslash E_1) + m(E_3 \backslash E_2) + \cdots m(E_n \backslash E_{n-1}) \cup \cdots) = m(E$$

- If $E_1 \supset E_2 \supset E_3 \supset \cdots$, $m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n) = \inf_{n=1,2,\ldots} m(E_n)$.

  Proof. $E_1 \backslash E_2, E_1 \backslash E_3, E_1 \backslash E_4, \ldots$, $m(E_1 \backslash (\bigcap_{n=1}^{\infty} E_n)) = \lim_{n\to\infty} m(E_1 \backslash E_n)$. $m(E_1) - m(\bigcap_{n=1}^{\infty} E_n) = m(E_1) - \lim_{n\to\infty} m(E_n)$.

- **Theorem (Caratheodory condition)**: The following are equivalent:

  (1) E is measurable.
  (2) $m^*(A) = m^*(A \cap E) + m^*(A \cap E^{\complement}), \forall A \subseteq \mathbb{R}^n$.

  Proof.

  (1) $\Rightarrow$ (2) Note that $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^{\complement})$. Need to show $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^{\complement})$. For the arbitrary A, pick $U \supseteq A$, U is open, $m^*(U) \leq m^*(A) + \epsilon$. Therefore, $m^*(A) + \epsilon \geq m^*(U) = m((U \cap E) \cup (U \cap E^{\complement})) = m(U \cap E) + m(U \cap E^{\complement}) \geq m^*(A \cap E) + m^*(A \cap E^{\complement})$.
  (2) $\Rightarrow$ (1) Assume $m^*(E) < \infty$. Pick $U \supseteq E$ such that $m^*(U) \leq m^*(E) + \epsilon$. Apply (2) with $A = U$, $m^*(U) = m^*(U \cap E) + m^*(U \cap E^{\complement}) = m^*(E) + m^*(U \backslash E)$. Given that $m^*(E) + \epsilon \geq m^*(U) = m^*(U \cap E) + m^*(U \cap E^{\complement}) = m^*(E) + m^*(U \backslash E)$ and $m^*(E) < \infty$, we can conclude that $m^*(U \backslash E) < \epsilon$.

- **Lemma**: If E, F satisfy condition (2), then $E \cap F$ satisfies (2). For the given E, write $E = \bigcup_{n=1}^{\infty} (E \cap B(n))$.

3 Lebesgue Integral

- **Riemann Integral**: Find a partition, $a \leq x_0 \leq x_1 \leq \cdots \leq x_n = b$, for each $(x_i, x_{i+1}]$, choose $x_i \leq x_i^* \leq x_{i+1}$, consider the Riemann sum $S_n = \sum f(x_i^*)|x_{i+1} - x_i| = S_n(P, x_i^*)$. If $\lim_{P_n \to \infty} S_n(P_n, x_i^*)$ exists to then f is Riemann integrable.
- Consider $\int \chi_{\mathbb{Q}(x)} dx$. We can choose $x_i^* \in \mathbb{Q}$, then $S_n(x_i) = 1$. While choose $x_i^* \notin \mathbb{Q}$, then $S_n(x) = 0$. Therefore, it is not Riemann integrable.
- Need to develop a new notion of integral such that
  - $\int \chi_E = m(E)$, if E is measurable.
  - $\int (\chi_E + \chi_F) = \int \chi_E + \int \chi_F = m(E) + m(F)$.
  - $\int C\chi_E = C \int \chi_E = Cm(E)$
  - $\int C_1\chi_{E_1} + C_2\chi_{E_2} + \cdots + C_n\chi_{E_n} = C_1 \int \chi_{E_1} + C_2 \int \chi_{E_2} + \cdots + C_n \int \chi_{E_n} = C_1 m(E_1) + C_2 m(E_2) + \cdots C_n m(E_n)$.
- **Definition**: We call f is simple if $f = C_1\chi_{E_1} + C_2\chi_{E_2} + \cdots + C_n\chi_{E_n}$.
- **Definition**: $\int f = \int C_1\chi_{E_1} + C_2\chi_{E_2} + \cdots + C_n\chi_{E_n} = C_1 m(E_1) + C_2 m(E_2) + \cdots C_n m(E_n)$, where $C_1, C_2, \cdots, C_n \geq 0$, $E_1, \ldots E_n$ are measurable.
- **Lemma**: if $C_1\chi_{E_1} + C_2\chi_{E_2} + \cdots + C_n\chi_{E_n} = C_1'\chi_{E_1'} + C_2'\chi_{E_2'} + \cdots + C_m'\chi_{E_m'}$, then $C_1 m(E_1) + C_2 m(E_2) + \cdots C_n m(E_n) = C_1' m(E_1') + C_2' m(E_2') + \cdots C_m' m(E_m')$.

3.1 Measurable function

- **Definition**: $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is measurable if $\{x : f(x) > \lambda\} = f^{-1}((\lambda, +\infty])$ is measurable for all $\lambda \in \mathbb{R} \cup \{\pm\infty\}$.
- **Lemma**: $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is measurable if and only if one of the following holds.
  - (a) $\{x : f(x) > \lambda\}$ is measurable for all $\lambda \in \mathbb{R} \cup \pm\infty\}$.
  - (b) $\{x : f(x) \leq \lambda\}$ is measurable for all $\lambda \in \mathbb{R} \cup \pm\infty\}$.
  - (c) $\{x : f(x) \geq \lambda\}$ is measurable for all $\lambda \in \mathbb{R} \cup \pm\infty\}$.
  - (d) $\{x : f(x) < \lambda\}$ is measurable for all $\lambda \in \mathbb{R} \cup \pm\infty\}$.
  - (e) $\{x : f(x) \in I\}$ is measurable for all open interval I.
  - (f) $\{x : f(x) \in U\}$ is measurable for all open set U.
  - (g) $\{x : f(x) \in D\}$ is measurable for all closed set D.

  Proposition: $f : X \to Y$, then for $A \subseteq Y$ and $B \subseteq Y$ we have $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, and $f^{-1}(A^\complement) = [f^{-1}(A)]^\complement$.

  Proof.

  (a) $\Leftrightarrow$ (b): $\{x : f(x) \leq \lambda\} = f^{-1}([-\infty, \lambda]) = f^{-1}((\lambda, \infty]^\complement) = [f^{-1}((\lambda, \infty])]^\complement$, since $f^{-1}((\lambda, \infty])$ is measurable, hence its complement is also measurable, which yields (b).

  (b) $\Rightarrow$ (c): $f^{-1}([\lambda, \infty]) = f^{-1}(\bigcap_{n=1}^{\infty} (\lambda - \frac{1}{n}, +\infty]) = \bigcap_{n=1}^{\infty} f^{-1}((\lambda - \frac{1}{n}, +\infty])$, hence $f^{-1}([\lambda, \infty])$ is measurable.

  (c) (c) $\Rightarrow$ (e): $I = (a, b) = [-\infty, b) \cap (a, \infty]$, hence $f^{-1}((a, b)) = f^{-1}([-\infty, b) \cap (a, \infty]) = f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty])$.

  (d) $\Rightarrow$ (f): Let $U = \bigcup_{i=1}^{\infty} (a_i, b_i) \in \mathbb{R}$, then U is an open set. Therefore, $f^{-1}(U) = f^{-1}(\bigcup_{i=1}^{\infty} (a_i, b_i)) = \bigcup_{i=1}^{\infty} f^{-1}((a_i, b_i))$. Hence, $f^{-1}(U)$ is measurable.

  (e) $\Rightarrow$ (g): $f^{-1}(D) = [f^{-1}(D^\complement)]^\complement$, since $D^\complement$ is open, then $f^{-1}(D^\complement)$ is measurable, therefore, $f^{-1}(D)$ is also measurable.

- **Lemma**: If f, g are measurable, then $f + g$, $f \cdot g$ are measurable.

  Proof. $\{f(x) + g(x) > \lambda\} = \{f(x) > \lambda - g(x)\} = \bigcup_{r \in \mathbb{R}} [\{g(x) \geq r\} \cap \{f(x) > \lambda - r\}]$. If $f(x) > \lambda - g(x)$, just take $r = g(x)$, then $f(x) > \lambda - r$. Next we need to make some adjustment. $\{f(x) + g(x) > \lambda\} = \{f(x) > \lambda - g(x)\} = \bigcup_{r \in \mathbb{Q}} [\{g(x) > r\} \cap \{f(x) > \lambda - r\}]$. If $f(x) > \lambda - g(x)$, just take $r < g(x)$, then $f(x) > \lambda - r$ and $r \in \mathbb{Q}$.

- **Lemma**: If $f_1, f_2, \ldots$ are measurable, then $\limsup_{n \to \infty} f_n$ and $\liminf_{n \to \infty} f_n$ exist.

  Proof. Consider $\{x : \limsup_{n \to \infty} f(x) > \lambda\} = \bigcup_{r \in \mathbb{Q}} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{f_n(x) > \lambda + r\}$. There are infinitely many of n such that $f_n(x) > \lambda + r$ for some $r > 0$.

- **Theorem**: If f is positive and measurable, then there is a sequence of positive simple function $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$ such that $\lim_{n \to \infty} f_n(x) = f(x), \forall x \in \mathbb{R}$. If f is bounded, the convergence can be chosen to be uniformly convergence.

  Proof. Pick $M, N \in \mathbb{N}$, consider intervals $[0, \frac{1}{N}), [\frac{1}{N}, \frac{2}{N}), \ldots, [M - \frac{1}{N}, M), [M, \infty]$, consider $E_1 = f^{-1}([0, \frac{1}{N})), E_2 = f^{-1}([\frac{1}{N}, \frac{2}{N})), \ldots, E_{MN} = f^{-1}([M - \frac{1}{N}, M)), E_{MN+1} = f^{-1}([M, \infty])$. Then let $f = 0 \cdot \chi_{E_1} + \frac{1}{N} \chi_{E_2} + \cdots + M \chi_{E_{MN+1}}$.

- $f \sim \sum_{k=0}^{MN} \frac{k}{N} \chi_{f^{-1}(k/N, (k+1)/N)}$ on $f^{-1}([0, M])$.
- **Definition**: $f : \mathbb{R}^n \to [0, +\infty]$, measurable. Then

$$\underline{\int} f = \sup\{\int g : 0 \leq g \leq f, g \text{ simple }\}, \text{ and } \overline{\int} f = \inf\{\int g : f \leq g, g \text{ simple }\}.$$

- **Some basic properties**:

(1) If f is simple, $\underline{\int} f = \overline{\int} f$.
(2) If $f \leq g$, $\int f \leq \int g$.
(3) $\int cf = c \int f$, for $0 \leq c \leq \infty$.
(4) $\int (f+g) = \int f + \int g$.
(5) $\int \min\{f(x), n\} dx \Rightarrow \int f$ as $n \to \infty$.
(6) $\int f \chi_{|x|<n} \to \int f$ as $n \to \infty$.

- **Monotone convergence theorem**: Assume $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$, and $f_n \to f$ a.e. (almost everywhere) , then $\int f_n \to \int f$, where $\int f < \infty$.

  Proof.

  (5) $\int \min\{f(x), n\} dx$ is an increasing sequence. For some N, we have $\int \min\{f(x), n\} dx \sim \int f$, $\exists g = c_1 \chi_{E_1} + c_2 \chi_{E_2} + \cdots + c_n \chi_{E_n}$ such that $\int g \geq \int f - \epsilon$, known that $g \leq f$. We may assume $0 \leq c_i \leq \infty$, then

  $$\int \min\{g(x), n\} \leq \int \min\{f(x), n\} \text{ for } n > c_1 + c_2 + \cdots + c_n \Leftrightarrow \int g(x) \sim \int f(x) \Rightarrow \int \min\{f(x), n\} \geq \int f(x).$$

  (6) Assume $f = \chi_E$ and recall that $\int \chi_E = m(E)$, then $\int \chi_E \chi_{|x|<n} = \int \chi_{E \cap \{|x|<n\}} = m(E \cap \{|x| < n\})$.

- **Proposition**: Suppose $E_1 \leq E_2 \leq \cdots \leq E_n \leq \cdots$, then

  $$m(\bigcup_{i=1}^{\infty} E_i) = \lim_{i \to \infty} m(E_i) = \sup_{i=1,\ldots} m(E_i).$$

  Suppose we have $F_1, F_2, \ldots, F_n, \ldots, F_i \cap F_j = \emptyset, \forall i \neq j$, then $m(F_1 \cup F_2 \cup \cdots \cup F_n \cup \cdots) = m(F_1) + m(F_2) + \cdots + m(F_n) + \cdots$, therefore, $E_n = E \cap \{|x| < n\}$ is an increasing set. Hence $\int \chi_E = m(E) = m(\bigcup_n^{\infty} E_n) = \lim_{n \to \infty} m(E_n)$. Then, $\int f = \underline{\int} f = \overline{\int} f$, $\underline{\int} f + g \geq \underline{\int} f + \underline{\int} g$, $\overline{\int} f + g \leq \overline{\int} f + \overline{\int} g$.

- **Lemma**: Let $f : \mathbb{R}^n \to (0, +\infty)$ measurable, $m(\{x : f(x) \geq \lambda) \leq \frac{1}{\lambda} \int f(x) dx$.

  Proof. Consider $\lambda \chi_{\{f(x) \geq \lambda\}} \leq f$, integrating both sides with respect to x yields

  $$\int \lambda \chi_{\{f(x) \geq \lambda\}} = \lambda \int \chi_{\{f(x) \geq \lambda\}} = \lambda m(\{f(x) \geq \lambda\}) \leq \int f \Rightarrow m(\chi_{\{f(x) \geq \lambda\}}) \leq \frac{1}{\lambda} \int f.$$

- **Corollary**: $f : \mathbb{R}^n \to [0, +\infty]$ measurable, then $\int f = 0 \Leftrightarrow f(x) = 0$ a.e , i.e. $m(f^{-1}([0, +\infty])) = 0$.

  Proof. ($\Leftarrow$) This direction is trivial.

  ($\Rightarrow$) Pick $\lambda = \frac{1}{\lambda}$, then $m(\{x : f(x) \geq \frac{1}{n}) \leq n \int f = 0$. That is, $m(\{x : f(x) > 0\}) = m(\bigcup_{n=1}^{\infty} \{x : f(x) \geq \frac{1}{n}) = 0$.

- **Definition**: Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$, $f = f_+ - f_-$, where

  $$\begin{cases} f_+(x) = \max\{f(x), 0\}, \\ f_-(x) = \max\{-f(x), 0\}. \end{cases}$$

- **Definition**: f is integrable, write as $f \in L^1(\mathbb{R}^n)$ if $\int f_+ < \infty$, $\int f_- < \infty$. In this case,

  $$\int f = \int f_+ - \int f_-.$$

- **Lemma**: $f \in L^1$ iff $\int |f| < \infty$.

  Proof. $f = f_+ - f_-$, $|f| = f_+ + f_-$. If $f \in L^1$, then $\int f_+ < \infty$ and $\int f_- < \infty$, then $\int |f| = \int f_+ + f_- < \infty$. If $\int f < \infty$, then $\int f_+ + \int f_- < \infty \Rightarrow \int f_+ < \infty, \int f_- < \infty$, that is, $\int |f| < \infty$.

- Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, $f \in L^1(\mathbb{R}^n)$, $f < \infty$ and $\int f < \infty$, i.e. $f = f_+ - f_-$, where

$$\begin{cases} f_+(x) = \max\{f(x), 0\}, \int f_+ < \infty, \\ f_-(x) = \max\{-f(x), 0\}, \int f_- < \infty. \end{cases}$$

- **Definition**: If $f \in L^1(\mathbb{R}^n)$, then $\|f\| = \int |f|$. Besides, we have

$$\|f\|_p = \left(\int |f|^p \right)^{1/p}, \forall 1 \le p \le \infty.$$

  For example,

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p = \inf\{M : f(x) < M, \text{ a.e.}\}.$$

- **Theorem**: Let $f \in L^1(\mathbb{R}^n)$, let $\epsilon > 0$.
  (1) There is an (integrable) simple function $cg \subset c_1\chi_{E_1} + c_2\chi_{E_2} + \cdots + c_n\chi_{E_n}, c_i \in \mathbb{R}$ such that $\|f - g\|_1 < \epsilon$.
  (2) There is a continuous function g with compact support such that $\|f - g\|_1 < \epsilon$. Note that compact support means

$$\text{supp}(g) = \{x : g(x) \ne 0\}.$$

  (3) By (1), we may assume $f \in c_1\chi_{E_1} + c_2\chi_{E_2} + \cdots + c_n\chi_{E_n}$ with $c_i \ne 0$ and $m(E_i) < \infty, 1 \le i \le \infty$. Then we may assume further then $c_i \ne 0$ and E_i are bounded.

$$E_{i,n} = E_i \cap B_n, E_i = \bigcup_{n=1}^{\infty} E_{i,n}, m(E_i) = \lim_{n\to\infty} m(E_{i,n}),$$

  where B_n is a ball centered at 0. Then, we may assume further that $f = \chi_E$, we only need to show (2) for $f = \chi_E$, where E is bounded. Let

$$\begin{cases} g(x) = 1, & \text{if } \text{dist}(x, E) = 0, \\ g(x) = 0, & \text{if } \text{dist}(x, E) > \delta. \end{cases}$$

  Assume E is closed, $g(x) = 1 - \frac{1}{\delta}\text{dist}(x, E)$. Then,

$$0 \le g - f = \begin{cases} 0 \text{ if } \text{dist}(x, E) = 0 \text{ or } \text{dist}(x, E) > \delta, \\ (0, 1) \text{ otherwise.} \end{cases}$$

  Need to show that $m(\{x : 0 < \text{dist}(x, E) < \delta\}) \to 0$ if $\delta \to 0$.

$$\int |f - g| \le \int \chi \le m(\{x : 0 < \text{dist}(x, E) < \delta\}) < \epsilon.$$

  For a general E, pick a closed $D \subseteq E$ such that $m(E \backslash D) < \frac{\epsilon}{2}$.

- **Lemma**: If $f \in L^1(\mathbb{R}^n)$ and $\epsilon > 0$, there is continuous function g such that $\|f - g\|_1 \le \epsilon$. What about $\{x : f(x) = g(x)\}$? $m(\{x : f(x) \ne g(x)\})$ might be large.

- **Theorem (Lusin Theorem)**: Let $f \in L^1(\mathbb{R}^n)$, $\epsilon > 0$, then there is a measurable set $E \subseteq \mathbb{R}^n$ such that f_E is continuous (on E) and $m(\mathbb{R}^n \backslash E) < \epsilon$.

  Proof. Recall the Markov inequality. Assume $n = f - g > 0$, then $m(\{x : h(x) \ge \epsilon) \le \frac{1}{\epsilon}\int h$. For abitrary $k \in \mathbb{N}$, choose a continuous function g_k such that $\int |f - g_k| < \frac{\epsilon}{k^3}$. Then consider $\{x : |f - g_k| \ge \frac{1}{k}\} = F_k$. Then $m(F_k) \le \frac{1}{k}\int |f - g_k| \le k \cdot \frac{\epsilon}{k^3} = \frac{\epsilon}{k^2}$. Set $F = \bigcup_{k=1}^{\infty} F_k$. Then $m(F) \le \sum_{k=1}^{\infty} m(F_k) \le \sum_{k=1}^{\infty} \frac{\epsilon}{k^2} \le \epsilon$. So $g_k|_E$ converge uniformly to $f|_E$. Since $g_k|_E$ are continuous, so is $f|_E$.

- **Lemma**: Let $f \in L^1(\mathbb{R}^n)$, $\forall\epsilon$, there is a continuous function g with compact support such that $\|f - g\|_1 = \int |f - g| < \epsilon$.

- **Theorem (Erogorou's theorem)**: Let $f_n : D \to \mathbb{R}$ be measurable and $m(D) < \infty$ such that $f_n(x) \to f(x)$ a.e.. Then $\forall\epsilon > 0$, there is $E \subseteq D$ such that $f_n|_E$ converges to $f|_E$ uniformly and $m(D \backslash E) < \epsilon, \forall\epsilon > 0$.

- **Example**: $f_n = x^n$ on $[0,1]$, $f_n(x) \to f(x) = \begin{cases} 1, & x = 1 \\ 0, & 0 \le x < 1. \end{cases}$. But x^n converges to 0 uniformly on $[0, 1-\epsilon]$. Then $m(\{x : f_n(x)\})$ does not converge to $m(\{x : f(x)\}) = 0$, which implies that $\{x : |f_n(x) - f(x)| > \frac{1}{m}$ for some $n > N\}$. Then $0 = m(\bigcup_{m=1}^{\infty} \bigcap_{N=0} \infty\{x : |f_n(x) - f(x)| > \frac{1}{m},$ for some $n > N\})$. Then $0 = m(\bigcap_{N=1} \bigcup_{n=N}^{\infty} \{x : |f_n(x) - f(x)| > \frac{1}{m})$. That is, $\forall m, 0 = \lim_{N\to\infty} m(\{x : |f_n(x) - f(x)| > \frac{1}{m},$ for some $n > N\})$. That is equivalent to $\forall m, m(\{x : |f_n(x) - f(x)| > \frac{1}{m},$ for some $n > N\}) \to 0$. Hence, $E^{\complement} = \{x : |f_n(x) - f(x)| \le \frac{1}{m},$ for all $n > N\}$. Then there is N_m such that $m(\{x : |f_n(x) - f(x)| > \frac{1}{m},$ for some $n > N_m\}) < \frac{\epsilon}{2^m}$. Then $F = \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} F_m$, $m(F) = m(\bigcup_{m=1}^{\infty} F_m) \le \epsilon$. Now consider $E = F^{\complement} = \bigcap_{m=1}^{\infty} F_m^{\complement} = \bigcap_{m=1}^{\infty} \{x : |f_n(x) - f(x)| \le \frac{1}{m}, \forall n > N\}$. Notice: if $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots$, $\lim_{n\to\infty}(E_n) = m(\bigcap_{i=1}^{\infty} E_i)$ if $E_1 < \infty$. If $E_1 = \infty$, then it doesn't hold. Counterexample: let $E_n(n, +\infty)$, then $\lim_{n\to\infty} E_n = \infty$, $m(\bigcap_{i=1}^{\infty} E_n) = 0$.

4 Abstract Measure Space

- **Definition**: Boolean algebra. Let X be a set. B is a collection of subsets of X. B is called *Boolean algebra* if

  1) $\emptyset \in B$.
  2) If $E \in B$, then $E^{\complement} \in B$.
  3) If $E, F \in B$, then $E \cup F \in B$.

- **Definition**: σ-algebra. Let X be a set. B is a collection of subsets of X. B is called *σ-algebra* if

  1) $\emptyset \in B$.
  2) If $E \in B$, then $E^{\complement} \in B$.
  3) If $E_1, E_2, \ldots, E_n, \ldots \in B$, then $\bigcup_{n=1}^{\infty} E_i \in B$.

- **Definition**: The pair (X, B) is called a measurable space.

- **Example**:

  – $(X, \{\emptyset, X\})$ is a σ-algebra.
  – $(X, 2^X)$ is a σ-algebra.
  – $(\mathbb{R},$ elementary sets $)$ is a Boolean algebra.

- Recall that a measure space is (X, B, μ) where B is a σ-algebra, i.e. 1) $\emptyset \in B$.

  2) If $E \in B$, then $E^{\complement} \in B$.
  3) If $E_1, E_2, \ldots, E_n, \ldots \in B$, then $\bigcup_{n=1}^{\infty} E_i \in B$. mu is a measure i.e. $\mu(B) \in [0, +\infty]$,
  4) $\mu(\emptyset) = 0$.
  5) $\mu(\bigcup\limits_{n=1}^{\infty} E_n) = \sum\limits_{n=1}^{\infty} \mu(E_i), E_i \in B, E_i \cap E_j = \emptyset, i \ne j$.

- **Example**:

  0) Lebesgue measure space: $(\mathbb{R}^d, \mathcal{M}(\mathbb{R}^d), \mu)$ where $\mathcal{M}(\mathbb{R}^d)$ is Lebesgue measurable sets, and μ is Lebesgue measure.
  1) $(X, \{\emptyset, X\}, \mu \equiv 0)$.
  2) $(X, \{\emptyset, X\}, \mu)$, where mu is $\mu(\emptyset) = 0$ and $\mu(X) = S \in [0, +\infty]$.
  3) (X, B, μ) where $\mu \equiv 0$ or $\mu(E) = \begin{cases} 0, & E = \emptyset, \\ +\infty, & E \ne \emptyset \end{cases}$.
  4) $(X, 2^X, \mu)$, where $\mu(E) = \begin{cases} |E|, & \text{if } |E| < \infty, \\ \infty, & \text{otherwise.} \end{cases}$.
  5) $X, B = E \subseteq X, E$ is countable or $E^{\complement}$ is countable$\}, \mu(E) = \begin{cases} 0, & E \text{ is countable,} \\ S, & \text{otherwise.} \end{cases}$
  6) **Dirac measure** (X, B, σ_{x_0}), where $\sigma_{x_0}(E) = \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E. \end{cases}$

7) **Definition**: let $\mathcal{C}$ be a collection of subsets of X. The σ-algebra generated by $\mathcal{C}$ is the smallest σ-algebra contain $\mathcal{C}$ ($\langle \mathcal{C} \rangle$), where $\langle \mathcal{C} \rangle = \bigcap_{B \supseteq \mathcal{C}} B$, B is a σ-algebra.

- **Lemma**: $B_\lambda, \lambda \in \Lambda$. $B\lambda$ is a σ-algebra of X. Then $B = \bigcap\limits_{\lambda \in \Lambda} B_\lambda$ is a σ-algebra.

  Proof. Need to show

  1) $\emptyset \in B$.
  2) If $E \in B$, then $E^{\complement} \in B$.
  3) If $E_1, E_2, \ldots, E_n, \ldots \in B$, then $\bigcup_{n=1}^{\infty} E_i \in B$.
  4) If $\emptyset \in B_\lambda, \forall \lambda \Rightarrow \emptyset \in \bigcap\limits_{\lambda \in \Lambda} B_\lambda = B$.
  5) If $E \in B = \bigcap\limits_{\lambda \in \Lambda} B_\lambda$, then $E \in B_\lambda, \forall \lambda \Rightarrow E^{\complement} \in B_\lambda \Rightarrow E^{\complement} \in \bigcap\limits_{\lambda \in \Lambda} B_\lambda$.
  6) Same.

- **Borel-algebra**: let X be a topological space, $\tau = \{\text{open sets}\}$. Then $\{\tau\}$ is called Borel-algebra.
- **Definition**: (X, B, μ) is complete if any subset of a null set (such a set is measurable with measure 0) is null.
- **Example**: Lebesgue measure space $(\mathbb{R}^d, \mathcal{M}(\mathbb{R}^d), \mu)$ is complete, however, $(\mathbb{R}^d, \text{Borel}(\mathbb{R}^d), \mu)$ is not complete.
- **Remark**: one can complete any measure space.
- **Definition**: $f : X \to \mathbb{R}$ is measurable if $f^{-1}((r, +\infty]) \in B, \forall r \in \mathbb{R}$.
- **Lemma**: Let $f : X \to [0, +\infty]$ be measurable, then there is a sequence of simple functions $0 \le f_1 \le f_2 \le f_3 \le \cdots f_n \le \cdots \le f$ such that $f_n(x) \to f(x)$ as $n \to \infty$. If f is bounded, the convergence can be chosen to be uniform convergence. $f_n = \sum\limits_i t_{i-1} \chi_{f^{-1}([t_{i-1}, t_i))}$.
- **Definition**: Let $f : X \to [0, +\infty]$ be measurable. Define $\int f d\mu = \sup\{\int g : 0 \le g \le f, g \text{ simple}\}$ if $g = C_1 \chi_{E_1} + C_2 \chi_{E_2} + \cdots + C_n \chi_{E_n}$.
- **Some properties of the above integral**:

  1) $\int C_1 f + C_2 g d\mu = C_1 \int f d\mu + C_2 \int g d\mu$.
  2) $\int \min\{f, n\} d\mu \to \int f d\mu$.
  3) If $E_1 \subseteq E_2 \subseteq \cdots E_n \subseteq X$, $\bigcup\limits_{i=1}^{n} E_i = X$, then $\int f \chi_{E_n} d\mu \to \int f d\mu$. If $f = \chi_F$, then $\in f \chi_{E_n} d\mu = \int \chi_F \chi_{E_n} d\mu = \int \chi_{F \cap E_n} d\mu = m(F \cap E_n)$. Therefore, $(F \cap E_1) \subseteq (F \cap E_2) \subseteq \cdots (F \cap E_n) \subseteq (F \cap X)$, then $\bigcup_{i=1}^{n} (F \cap E_i) = F$.

- Consider $f : X \to [-\infty, +\infty]$ measurable on $(X, B\mu)$, define $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Then we have the decomposition of f: $f = f_+ - f_-$. $f \in L^1(\mu)$ if and only if $\begin{cases} \int f_+ d\mu < \infty, \\ \int f_- d\mu < \infty \end{cases}$, in other words, $\int |f| d\mu < \infty$.
- **Example**: $X = \mathbb{N}$, $B = 2^{\mathbb{N}}$, $\mu(E) = |E|$, $f : \mathbb{N} \to \mathbb{R}$, then $f \in L^1(\mu) = \{f : X \to \mathbb{R}, f = 0 \text{ except a countable subsets}\}$. In other words, f is supported by a countable subset. Then $\sum\limits_{f(x) \neq 0} |f(x)| < \infty$.
- **Example**: $X, B, x_0 \in X$, $\sigma_{x_0}(E) = \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{otherwise} \end{cases}$, $f : X \to \mathbb{R}$ is measurable. $\int f d\sigma_{x_0} = f(x_0)$, $f = \chi_E$, $int \chi_E d\delta_{x_0} = \delta_{x_0(E} = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{otherwise} \end{cases} = \chi_E(x_0)$.
- **Example**: X, E is the countable subsets i.e. $E = \{E$ is countable or $X \backslash E$ is countable. $\mu(E) = |E|$. What does the functions in $L^1(\mu)$ look like?
- **Definition**: (X, B, μ) is finite measure space if $\mu(E) < \infty$ is a probability space if $\mu(E) = 1$.

- **Theorem (Egorov's theorem)**: Let (X, B, μ) be a measurable space. Let $f_n(x) \to f(x)$, a.e. Then for any $\epsilon > 0$, there is $E \in X$ such that $\mu(E) < \epsilon$. Then $f_n|_{X \setminus E}$ converges to $f|_{X \setminus E}$.

- The convergence theorem: If $f_n(x) \to f(x)$a.e. when do we have $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$. In other words, when can you switch the limit operation with the integration, i.e. $\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu = \int f d\mu$. In general, this is not true. For instance, let $f_n(x) = \begin{cases} 1, & \text{if } x \in [n, n+1), \\ 0, & \text{elsewhere} \end{cases}$, we have $\forall x$, $\lim_{n\to\infty} f_n(x) d\mu = 0$ and $\int f_n d\mu = 1, \forall n$. However, $\int \lim_{n\to\infty} f_n d\mu = \int 0 d\mu = 0$.

- **Theorem (Monotone convergence theorem)**: Let (X, B, μ) be a measurable space and consists of $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$, then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$. Space case: If $f_n = \chi_{E_n}$, then $E_n \subseteq E_{n+1}$ and $\lim_{n\to\infty} f_n = \chi_{E_n}$, $\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu = \mu(\bigcup_{n=1}^{\infty} E_n)$.

  Proof. It is clear that $\lim_{n\to\infty} \int f_n d\mu \le \int f d\mu$. Let's prove $\lim n \to \infty \int f_n d\mu \ge \int f d\mu$. Pick $g \le f$, where $g = C_1 \chi_{E_1} + C_2 \chi_{E_2} + \cdots + C_m \chi_{E_m}, C_i > 0, i = 1, \ldots, m, E_i \cap E_j = \emptyset, i \ne j$ is a simple function. Then $\int f d\mu \sim \int g d\mu$. Consider $(1-\epsilon)g$ and consider $E_{i,n} := \{x \in E_i, f_n(x) > (1-\epsilon)C_i\} \subseteq E_i$. Since $f_n \le f_{n+1}$, $E_{i,n} \subseteq E_{i,n+1}$ since $f_n(x)$ converges monotonically to $f(x)$. $\bigcup_{n=1}^{\infty} E_{i,n} = E_i$. Compare $f_n > \sum_{i=1}^{m} (1-\epsilon) C_i \mu(E_{i,n})$, $\lim_{n\to\infty} \int f_n d\mu \ge \sum_{i=1}^{m} (1-\epsilon) C \lim_{n\to\infty} \mu(E_{i,n}) = \sum_{i=1}^{m} (1-\epsilon) C_i \mu(E_i) = (1-\epsilon) \int g d\mu \sim \int f d\mu$.

- **Lemma (Fatou's lemma)**: Let (X, B, μ) be a measure space and let $f_1, f_2, \cdots, f_n$, $f_i : X \to [0, +\infty]$ be a sequence of measurable functions. Then $\int \liminf_{n\to\infty} f_n d\mu \le \liminf_{n\to\infty} \int f_n d\mu$.

  Proof. Write $F_N = \inf_{n>N} f_n$, $\int \liminf_{n\to\infty} f_n = \int \lim_{N\to\infty} F_N d\mu$. Notice that $F_N \le F_{N+1}$, then $\int \lim_{N\to\infty} F_N d\mu = \lim_{N\to\infty} \int f_N d\mu \le \liminf_{n\to\infty} f_n$.

- **Convergence theorems**

  1) Monotonic convergence theorem: For $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$, measurable function, $f_n \to f$ as $n \to \infty$. Then

  $$\int \lim_{n\to\infty} f_n(x) d\mu = \lim_{n\to\infty} \int f_n(x) d\mu.$$

  2) Fatou's lemma: for $f_1, f_2, \ldots, f_i : X \to [0, +\infty], i = 1, 2, \ldots$ such that $f_n \to f$ a.e. Then

  $$\int \liminf_{n\to\infty} f_n(x) d\mu \le \liminf_{n\to\infty} \int f_n(x) d\mu.$$

  3) Dominated convergence theorem: Let $f_1, f_2, \ldots, f_i : X \to [0, +\infty], i = 1, 2, \ldots$ such that $f_n \to f$ a.e. Suppose there is $g \in L^1(X)$ such that $|f_i(x)| \le g(x)$, in particular $f(x) \le g(x)$. Then,

  $$\lim_{n\to\infty} \int f_n(x) d\mu = \int f(x) d\mu.$$

  Proof. Consider $f_n + g$. Then $f_n + g$ positive and $f_n + g \to f + g$, by Fatou's lemma, we have

  $$\int \liminf_{n\to\infty} (f_n(x) + g(x)) \le \liminf_{n\to\infty} \int f_n(x) + g(x).$$

  Since the $f_n \to f$ as $n \to \infty$, then $\liminf_{n\to\infty}(f_n + g) = f + g$. Then the above becomes

  $$\int \liminf_{n\to\infty} (f_n(x) + g(x)) = \int f(x) + g(x) = \int f(x) + \int g(x) \le \liminf_{n\to\infty} \int (f_n(x) + g(x)) = \liminf_{n\to\infty} \int f_n + \int g(x),$$

  yielding

  $$\int f(x) \le \liminf_{n\to\infty} \int f_n.$$

Now consider $g - f_n$, which is positive and $g - f_n \to g - f$ as $n \to \infty$. Then, again, by Fatou's lemma, we have the following

$$\int \liminf(g-f_n) = \int g-f \leq \liminf_{n\to\infty} \int (g-f_n) \leq \liminf_{n\to\infty}(\int g-\int f_n) = \int g+\liminf_{n\to\infty}(-\int f_n) = \int g-\limsup_{n\to\infty} \int f_n.$$

Therefore, $\int f \geq \limsup_{n\to\infty} \int f_n$. Above all, we can obtain

$$\liminf_{n\to\infty} \int f_n \geq \int f \geq \limsup_{n\to\infty} f_n.$$

and by $\liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} f_n$, we have

$$\liminf_{n\to\infty} \int f_n = \limsup_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n.$$

The proof is complete.

- **Modes of convergence**: Let $f_1, f_2, \ldots, f_n, \ldots$ be real-valued measurable functions on (X, B, μ). Then we define

  1) $f_n \to f$ a.e.
  2) $f_n \rightrightarrows f$ uniformly (or we say $f_n \to f$ in L^∞) , if $\forall \epsilon > 0, \exists N_\epsilon$ such that $|f_n(x) - f(x)| < \epsilon, \forall x, \forall n > N_\epsilon$. Notice $\|f\|_\infty = \sup\{|f(x)|"x \in X\}$, then $f_n \rightrightarrows f$ iff $\|f_n - f\|_\infty \to 0$.
  3) f_n converges almost uniformly to f if $\forall\epsilon > 0$, there is $E \in B$ such that $\mu(E) < \epsilon$ and $f_n|_{E^\complement} \rightrightarrows f|_{E^\complement}$.
  4) $f_n \to f$ in L^1 if $\int |f_n - f| \to 0$ as $n \to \infty$.
  5) $f_n \to f$ in measure if $\forall\epsilon > 0$, then $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$.

  Relationships between these convergences. 2) implies 1), 2) implies 3), 3) implies 5), 4) implies 5) and 3) implies 1).

  Proof of 3) implies 5). Need to show that $\forall\epsilon' > 0$, $\exists N$ such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon', \forall n > N$. By 3), $\exists E$ such that $m(E) < \epsilon'$ and $f_n|_{E^\complement} \rightrightarrows f|_{E^\complement}$. Then $\exists N$ such that

  $$|f_n(x) - f(x)| < \epsilon, \forall x \in E^\complement, n > N.$$

  Then

  $$\forall n > N, \{x : |f_n(x) - f(x) \geq \epsilon\} \subseteq E \Rightarrow \mu(\{x : |f_n(x) - f(x) \geq \epsilon\}) \leq \mu(E) \leq \epsilon'.$$

  The proof is complete.

- **Modes of convergence**:

  1) Pointwisely convervence (a.e.).
  2) Uniform convergence.
  3) Almost uniform convergence.
  4) L^1 convergence.
  5) Convergence in measure: $f_n \to f$ in measure iff $\forall\epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$.

  As for 1) and 3), if $\mu(X) < \infty$, then by Egorov theorem, we have 1) $\Rightarrow$ 3).

- **Preposition**:

  1) If $f_n \to f$ (in some sense of above modes of convergence) and $f_n \to g$ (in the same sense), we have $f = g, a.e.$
  2) If $f_n \to f$ (in some sense of above modes of convergence) and $f_n \to g$ (in a different sense), we still have $f = g, a.e.$

- **Lemma**: If $f_n \to f$ pointwisely a.e. and $f_n \to g$ in measure a.e., then $f = g$ a.e.

  Proof. $\forall\epsilon > 0$, consider $A = \{x : |f(x) - g(x)| < \epsilon\}$. Let us show that $\mu(A_\epsilon) = 0$. Then $\{x : f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$ has measure zero. Consider $A_{N,\epsilon} = \{x \in A_\epsilon, |f_n(x) - f(x)| < \frac{\epsilon}{2}, \forall n > N\}$. Then

  $$\bigcup_{N=1}^{\infty} A_{N,\epsilon} = A_\epsilon$$

  up to a null set. Assume $\mu(A_\epsilon)$, then $\mu(A_{N,\epsilon}) > 0$ for some N. Consider $A_{N,\epsilon}$, if $x \in A_{N,\epsilon}$, $|f(x) - g(x)| > \epsilon$, since $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Then $f_n(x) - g(x) = |f(x) - g(x) - [f(x) - f_n(x)]| \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$. But $f_n \to g$ in measure, $\mu(\{x : |f_n(x) - g(x)| \geq \frac{\epsilon}{2}\}) \geq \mu(A_{N,\epsilon}) > 0$, a contradiction.

- **Theorem**: If $f_n \to f$ in measure then there is a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k} \to f \text{ pointwisely } a.e.$$

Proof. Pick $\{n_k\}$ such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^k}\}) \leq \frac{1}{2^k}.$$

Let us show that $f_{n_k} \to f$ a.e. as $k \to \infty$. By the subsequent remark, we have the set of x such that $f_{n_k}(x) \not\to f(x)$ is

$$\bigcup_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}.$$

We only need to show that $\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}$ is null. Next we only need to show $\bigcup_{n=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}$ is arbitrarily small for large enough $N > 1$. Choose $k > m$,

$$\mu(\bigcup_{k=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}) \leq \mu(\bigcup_{k=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^k}) \leq \sum_{k=N}^{\infty} \frac{1}{2^k} = \frac{1}{2^{N-1}}.$$

- **Remark**: If $f_n \to f$, the set of divergent points is $|f_n(x) - f(x)| > \frac{1}{2^m}$.
- **Theorem**: If $f_n \to f$ in measure then there is a subsequence $\{f_{n_k}\}$ such that

$$\forall \epsilon > 0, \exists E, \text{ such that } \mu(E) < \epsilon \text{ and } f_{n_k}|_{E^\complement} \rightrightarrows f|_{E^\complement}.$$

Proof. By De Morgan's laws, we have

$$\left[\bigcup_{k=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}\right]^\complement = \bigcap_{k=N}^{\infty} \{x : |f_{n_k}(x) - f(x)| \leq \frac{1}{2^m}\}.$$

For each m, choose N_m such that

$$\bigcup_{k=N_m}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}\} < \frac{\epsilon}{2^m}.$$

Consider $E = \bigcup_{m=1}^{\infty} \bigcup_{k=N_m}^{\infty} \{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^m}\}$, then $\mu(E) \leq \epsilon$. Therefore,

$$E^\complement = \bigcap_{m=1}^{\infty} \bigcap_{k=N_m}^{\infty} \{x : |f_{n_k}(x) - f(x)| \leq \frac{1}{2^m}\}.$$

- **Theorem (Ergorov)**: Let (X, B, μ) be a finite measure space, then

$$f_n \to f \text{ a.e. iff } f_n \to f \text{ almost everywhere.}$$

4.1 Differentiation theorem

- **Definition**: f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. Furthermore, f is differentiable if $f'(x)$ exists everywhere.
- **Fundamental theorem of calculus**:
  1) Let $f : [a, b] \to \mathbb{C}$ be continuous, consider

  $$F(x) = \int_a^x f(t)dt.$$

  Then $F'(x) = f(x)$.
  2) Let $F : [a, b] \to \mathbb{R}$. Then

  $$\int_a^b F'(t)dt = F(b) - F(a).$$

4.2 Lebesgue differentiation in $\mathbb{R}$

- **Theorem**: Let $f \in L^1(\mathbb{R})$. Define

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{\mathbb{R}} f(t)\chi_{[-\infty,x]}(t)dt.$$

Then F is continuous and differentiable almost everywhere and $F'(x) = f(x)$ a.e. and

$$F'(x) = f(x) a.e.$$

Proof. *Continuity*: let $x_0 \in \mathbb{R}$,

$$F(x_0+h)-F(x_0) = \int_{\mathbb{R}} [\chi_{(-\infty,x_0+h)}(t)f(t)-\chi_{(-\infty,x_0)(t)}f(t)]dt = \int_{\mathbb{R}} \chi_{[x_0,x_0+h]}(t)f(t)dt \overset{\text{Dominated Convergence Theorem}}{\rightarrow} 0 \text{ as } h \to 0.$$

Differentiability: In other words,

$$\lim_{h\to 0^+} \frac{1}{h}\int_{x}^{x+h} f(t)dt = f(x) a.e. \text{ and } \lim_{h\to 0^+} \frac{1}{h}\int_{x-h}^{x} f(t)dt = f(x) a.e.$$

If $g \in C_c(\mathbb{R})$ (continuous with compact support), then the theorem holds. In general, $\forall \epsilon > 0, \exists g \in C_c(\mathbb{R})$ such that

$$\int |f-g|d\mu < \epsilon \text{ or } \|f-g\|_{L^1} < \epsilon.$$

Hope to use g as a bridge!

- **Lemma** Let $f \in L^1(\mathbb{R})$ and let $\lambda > 0$. Then,

$$m(\{x : \sup_{h>0} \frac{1}{h}\int_{x}^{x+h} |f(t)|dt \geq \lambda\}) \leq \frac{1}{\lambda}\int_{\mathbb{R}} |f(t)|dt.$$

- Now let's prove that

$$\lim_{h\to 0^+} \frac{1}{h}\int_{x}^{x+h} f(t)dt = f(x) a.e.$$

For any $\epsilon > 0$, choose $g \in C_c(\mathbb{R})$ such that $\|f-g\|_{L^1} < \epsilon$. Fix $\lambda > 0$, by the above Lemma, we have

$$E^1_{\epsilon,\lambda} = m(\{x \in \mathbb{R}, \sup_{h>0} \frac{1}{h}\int_{x}^{x+h} |f(t)-g(t)|dt \geq \lambda\}) \leq \frac{\epsilon}{\lambda} \text{ and } E^2_{\epsilon,\lambda} = m(\{x \in \mathbb{R}, |f(x)-g(x)| \geq \lambda\}) \leq \frac{\epsilon}{\lambda}.$$

Let $E_{\epsilon,\lambda} := E^1_{\epsilon,\lambda} \cup E^2_{\epsilon,\lambda}$, $m(E_{\epsilon,\lambda}) \leq \frac{2\epsilon}{\lambda}$ and $\forall x \in (E_{\epsilon,\lambda})^{\complement}$, we have

$$\begin{cases} \frac{1}{h}\int_{x}^{x+h} |f(t)-g(t)|dt < \lambda \\ |f(x)-g(x)| < \lambda \end{cases}$$

Note that

$$\big|\frac{1}{h}\int_{x}^{x+h} g(t)dt - g(x)\big| < \lambda \text{ for sufficiently small } h.$$

Then, for any $x \in (E_{\epsilon,\lambda})^{\complement}$,

$$\begin{aligned} \big|\frac{1}{h}\int_{x}^{x+h} f(t)dt - f(x)\big| &= \big|\frac{1}{h}\int_{x}^{x+h} f(t)dt - \frac{1}{h}\int_{x}^{x+h} g(t)dt + \frac{1}{h}\int_{x}^{x+h} g(t)dt - g(x) + g(x) - f(x)\big| \\ &= \big|\frac{1}{h}\int_{x}^{x+h} f(t)dt - \frac{1}{h}\int_{x}^{x+h} g(t)dt\big| + \big|\frac{1}{h}\int_{x}^{x+h} g(t)dt - g(x)\big| + \big|g(x)-f(x)\big| \\ &\leq 3\lambda. \end{aligned}$$

Consider $E_\lambda = \bigcap_\epsilon E_{\epsilon,\lambda}$, $m(E_\lambda) = 0$. Then

$$\big|\frac{1}{h}\int_{x}^{x+h} f(t)dt - f(x)\big| < \lambda a.e.$$

- **Example**: Let $f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise .} \end{cases}$. Then

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & \text{if } x \leq 1, \\ 1, & \text{if } x \geq 2, \\ x-1, & \text{otherwise.} \end{cases}$$

- **Reading topics**:

  1) Bernoulli convolution
  2) Housdoff measure (dimension)
  3) IFS (Iterative Functions System)
  4) Banach - Tarschi paradox
  5) Amenable groups

- Now we consider higher dimensional case.

- **Theorem**: Let $f \in L^1(\mathbb{R}^d)$, then

$$\lim_{r \to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0 a.e.$$

  and

$$\lim_{r \to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy = f(x) a.e.$$

- Let $E \subseteq \mathbb{R}^d$, $0 < m(E) < \infty$, consider $f(x) = \chi_{E(x)}$.

$$\lim_{r \to 0} \frac{m(B(x,r) \cap E)}{m(B(x,r))} = \lim_{r \to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} \chi_E(y) dy = \chi_E(x) a.e.$$

  Consider 1-D case,

$$\lim_{|I| \to 0} \frac{m(I \cap E)}{m(I)} = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases} \quad a.e.$$

- **Theorem**: Any monotonic function is differentiable almost everywhere.

- **Theorem**: Any monotonic function is continuous almost everywhere.

- Assume F is increasing, if F is not continuous at $x = x_0$, then

$$\lim_{x \to x_0^-} F(x) \leq F(x_0) \leq \lim_{x \to x_0^+} F(x).$$

- **Lemma**: Let F be monotonic, then

$$F = F_c + F_d,$$

  where F_c is continuous, and F_d is a jump function.

- **Example**: Let u be a Borel measure on $\mathbb{R}$. Define

$$F(x) = \mu((-\infty, x)).$$

  Let $x_0 \in \mathbb{R}$, consider $(x < x_0)$,

$$F(x_0) - F(x) = \mu((-\infty, x_0)) - \mu((-\infty, x)) = \mu([x, x_0)) = \mu(\emptyset) = 0 \text{ as } x \to x_0.$$

  Then F is positive increasing semi-continuous. On the other hand, if F is a such function, then there is a Borel measure μ such that $F(x) = \mu((-\infty, x))$.

- **Theorem**

1) Let $F : [a, b] \to \mathbb{R}$ be increasing function. Then

$$\int_a^b F'(x)dx \leq F(b) - F(a).$$

2) If F is absolutely continuous, then

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Proof.

1) Let $f_n(x) = \frac{F(x+\frac{1}{n})-F(x)}{\frac{1}{n}}$, then $f_n(x) \to F'(x) a.e.$. We have the following

$$\begin{aligned}
\int_a^b F'(x)dx &= \int_a^b \liminf_{n\to\infty} f_n(x)dx \\
&\leq \liminf_{n\to\infty} \int_a^b f_n(x)dx \\
&= \liminf_{n\to\infty} n \int_a^b F(x+\frac{1}{n}) - F(x)dx \\
&= \liminf_{n\to\infty} n \left[\int_a^b F(x+\frac{1}{n})dx - \int_a^b F(x)dx\right] \\
&= \liminf_{n\to\infty} n \left[\int_{a+1/n}^{b+1/n} F(x)dx - \int_a^b F(x)dx\right] \\
&= \liminf_{n\to\infty} n \left[\int_b^{b+1/n} F(x)dx - \int_a^{a+1/n} F(x)dx\right]
\end{aligned}$$

We extend F by add another piece, $F : [b, b + \frac{1}{n}] \to F(b)$. Since

$$\int_b^{b+1/n} F(x)dx = \frac{1}{n}F(b)$$

and notice that

$$\int_a^{a+1/n} F(x)dx \geq \int_a^{a+1/n} F(a)dx = \frac{1}{n}F(a).$$

Therefore,

$$\begin{aligned}
\int_a^b F'(x)dx &\leq \liminf_{n\to\infty} \int_a^b f_n(x)dx \\
&\leq \liminf_{n\to\infty} n \left[\frac{1}{n}F(b) - \frac{1}{n}F(a)\right] \\
&= F(b) - F(a).
\end{aligned}$$

- **Example**:

1) Heaviside step function. Let

$$F(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}, F'(x) = 0 a.e.$$

then we have

$$\int_{-1}^1 F'(x)dx = \int_{-1}^1 0dx = 0 \leq F(1) - F(-1) = 1.$$

2) Cantor Stair function. $F'(x) = 0, a.e.$ but $F(1) = 1, F(0) = 0$.

- **Definition**: A real-valued function $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any intervals $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ with $\sum_{i=1}^{n} |a_i - b_i| < \delta$, we have
$$\sum_{i=1}^{n} |F(a_i) - F(b_i)| < \epsilon.$$

- **Lemma**: Let $f \in L^1(\mathbb{R})$, for any $\epsilon > 0$, there is $\delta > 0$ such that if $E \subseteq \mathbb{R}$, $m(E) \le \delta$, then
$$|\int_E f d\mu| \le \epsilon.$$

  Proof. The conclusion holds for simple functions. Hence, we pick $g = c_1\chi_{E_1} + c_2\chi_{E_2} + \cdots + c_n\chi_{E_n} \sim f \in L^1$ such that $|g - f| \le \frac{\epsilon}{2}$. Then
$$\int_E |f| d\mu = \int_E |g + f - g| d\mu \le \int_E |g| d\mu + \int_E |f - g| d\mu \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- **Definition**: Let λ, μ be measure on $(X, \mathcal{M})$, then $\lambda \ll \mu$ (absolutely continuous) if any μ is a null set, then λ is null. In other words, $\mu(E) = 0 \Rightarrow \lambda(E) = 0, \forall E \in \mathcal{M}$. Furthermore, $\lambda \perp \mu$ if $\exists E \in \mathcal{M}$ such that $\mu(E) = 0$ and $\lambda(E^\complement) = 0$.

- **Theorem (Lebesgue-Radom-Nikodym Theorem)**: Let λ, μ be measure on $(X, \mathcal{M})$ with μ σ-finite.
  1) $\lambda = \lambda_c + \lambda_s$ such that $\lambda_c \ll \mu$, $\lambda_s \perp \mu$.
  2) There is $h \in L^1(\mu)$ such that $\lambda_c(E) = \int_E h d\mu$. ($h = \frac{d\lambda_c}{d\mu} \Rightarrow d\lambda_c = h d\mu$).

- **Example**: Take μ as Lebesgue measure, $\lambda = \int_E dF$, $\lambda((-\infty, x)) = F(x) - F(-\infty)$. Then $dF = d\lambda_c + d\lambda_s = h d\mu + d\lambda_s$.

- Recall that $\lambda_c \ll \mu$ if and only if $\mu(E) = 0 \Rightarrow \lambda_c(E) = 0$ and $\lambda_s \perp \mu$ if and only if $E \in \mathcal{M}$ such that $\mu(E^\complement) = 0$ and $\lambda_s(E) = 0$.

- **Lemma** Let λ, μ be measures on $(X, \mathcal{M})$. Assume λ finite, then the following are equivalent.
  1) $\lambda \ll u$ ($\mu(E) = 0 \Rightarrow \lambda(E) = 0$).
  2) $\forall \epsilon > 0, \exists \delta > 0$ such that $\mu(E) < \delta \Rightarrow \lambda(E) < \epsilon (\forall E \in \mathcal{M})$.

- Recall that a function $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if $\forall \epsilon > 0, \exists > 0$ such that for any $(a_1, b_1), \dots, (a_n, b_n)$ with $\sum_{i=1}^{n} |b_i - a_i| < \delta$. One has
$$\sum_{i=0}^{n} |F(b_i) - F(a_i)| < \epsilon.$$

  Proof.
  1) $\Rightarrow$ 2) If $\mu(E) = 0$, then by 2) $\lambda(E) < \epsilon$ for $\forall \epsilon > 0$.
  2) $\Rightarrow$ 1) By contradiction, if 2) does not hold, $\exists \epsilon_0 > 0, \forall \delta > 0, \exists E$ such that $\mu(E) \le \delta$ but $\lambda(E) \ge \epsilon_0$ with $\delta_n = \frac{1}{2^n}$. $\exists E_n$ such that $\mu(E_n) \le \frac{1}{2^n}$, but $\lambda(E_n) \ge \epsilon_0$. Let's consider $E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_k$,
$$\mu(E) \le \mu(\bigcup_{n=k}^{\infty} E_k = \sum_{n=k}^{\infty} \mu(E_n) \le \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}} \Rightarrow \mu(E) = 0.$$
     Therefore, $\lambda(E) = \lim_{k\to\infty} \lambda(\bigcup_{n=k}^{\infty} E_n) \ge \epsilon_0$.

5 Outer measure, pre-measure

- **Definition**: Let X be a set, and outer measure is a map $\mu^* : 2^X \to [0, +\infty]$ such that
  1) $\mu^*(\emptyset) = 0$.
  2) $\mu^*(E) \leq \mu^*(F)$ if $E \subseteq F$.
  3) $\mu^*(E_1 \cup E_2 \cup \cdots E_n \cup \cdots) \leq \mu^*(E_1) + \mu^*(E_2) + \cdots + \mu^*(E_n) + \cdots$ for countably many $E_1, E_2, \ldots$
- **Example**:
  1) Lebesgue outer measure: $m^*(E) = \inf\{\sum_{n=1}^{\infty} |I_n| : E \subseteq \bigcup_{n=1}^{\infty} I_n\}$.
  2) Hausdorff outer measure: Fix $s > 0$,

$$H_s(E) = \lim_{\sigma \to 0} \left\{ \int \{ \sum_{n=1}^{\infty} [\mathrm{diam}(B_n)]^s : E \subseteq \bigcup_{n=1}^{\infty} B_n, \mathrm{diam}(B_n) \leq \delta \} \right\}$$

  If $s = n$, $H_s \sim m^{*,n}$. If $s = \dim(B_n)$, $\mathrm{diam}(B_n) \sim V(B_n)$. Take Cantor set for example, we have the Lebesgue measure $\sum |I_n| = 2^n \left(\frac{1}{3}\right)^n \to 0$ as $n \to \infty$. However, the Hausdorff measure depends on the value s,

$$H_s = 2^n \left(\frac{1}{3}\right)^s = \begin{cases} \infty & \text{if } s < \frac{\ln 2}{\ln 3} \\ 0 & \text{if } s > \frac{\ln 2}{\ln 3} \\ a & \text{if } s = \frac{\ln 2}{\ln 3} \end{cases}.$$

- **Lemma**: $\exists d$ such that

$$H_s(E) = \begin{cases} 0 \text{ if } s > d \\ \infty \text{ if } s < d \end{cases}$$

  where $d = H - \dim(E)$.
- How to determine the dimension of Hausdorff space? $\dim \mathcal{H}_s = \frac{\ln a}{\ln b}$, where a is the scaling number, and b is the number of mappings from the original set.
- Suppose (X, d) is complete metric space, $f : X \to X$ such that $\exists 0 < \gamma < 1$, $d(f(x), f(y)) \leq \gamma d(x, y), \forall x, y, \exists x_0 \in X$ such that $f(x_0) = x_0$.
- **Definition (Caratheodory condition)**: Let μ^* be an outer measure. A set $E \subseteq X$ is said to be Caratheodory measurable if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement}), \forall A \subseteq X$.
- **Theorem**: Let $B = \{E \subseteq X : E$ Caratheodory measurable$\}$, then
  1) B is a σ-algebra and
  2) the restriction of $\mu^* \neq 0$ then B is a measurable set.

  Proof.
  1) It's obvious that $\emptyset \in B$, and $X \in B$. Then the two terms of the right-hand side of $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement}) \forall A$ are same when E is either $\emptyset$ or X. Assume $E, F \in B \Rightarrow E \cup F \in B$, so do $E^{\complement}$ and $F^{\complement}$. Now let's take a look at

$$\begin{aligned}
\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^{\complement}) &= \mu^*(A \cap (E \cup F)) + \mu^*(A \cap E^{\complement} \cap F^{\complement}) \\
&= \mu^*(A \cap ((E \cap F^{\complement}) \cup (E^{\complement} \cap F) \cup (E \cap F))) + \mu^*(A \cap E^{\complement} \cap F^{\complement}) \\
&= \mu^*((A \cap E \cap F^{\complement}) \cup (A \cap E^{\complement} \cap F) \cup (A \cap E \cap F)) + \mu^*(A \cap E^{\complement} \cap F^{\complement}) \\
&\leq \mu^*(A \cap E \cap F^{\complement}) + \mu^*(A \cap E^{\complement} \cap F) + \mu^*(A \cap E \cap F) + \mu^*(A \cap E^{\complement} \cap F^{\complement}) \\
&= [\mu^*(A \cap E \cap F^{\complement}) + \mu^*(A \cap E \cap F)] + [\mu^*(A \cap E^{\complement} \cap F) + \mu^*(A \cap E^{\complement} \cap F^{\complement})] \\
&= \mu^*(A \cap E) + \mu^*(A \cap E^{\complement}) \\
&= \mu^*(A).
\end{aligned}$$

  Next, let's show that the other direction.

$$\mu^*(A) = \mu^*((A \cap (E \cup F) \cup (A \cap (E \cup F)^{\complement})) \leq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^{\complement}).$$

Therefore, $\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^{\complement})$. Then we can extend the above to finitely many sets $E_1, E_2, \ldots, E_N \in B$, we have

$$\mu^*(A) = \mu^*(A \cap (\bigcup_{n=0}^{N} E_n)) + \mu^*(A \cap (\bigcup_{n=0}^{N} E_n)^{\complement}), \forall A.$$

We want to show the above property holds for $\bigcup\limits_{n=1}^{\infty} E_n \in B$, we need to show

$$\mu^*(A) = \mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)) + \mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)^{\complement}), \forall A.$$

By subadditivity, we have

$$\mu^*(A) \leq \mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)) + \mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)^{\complement}), \forall A.$$

So we need to show the other direction. Since $\bigcup_{n=0}^{N} E_n \subseteq \bigcup_{n=0}^{\infty} E_n$, $(\bigcup_{n=0}^{N} E_n)^{\complement} \supseteq (\bigcup_{n=0}^{\infty} E_n)^{\complement}$. We can directly get that

$$\mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)^{\complement}) \leq \mu^*(A \cap (\bigcup_{n=0}^{N} E_n)^{\complement}).$$

Next, we just need to show that

$$\mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)) \leq \lim_{N \to \infty} \mu^*(A \cap (\bigcup_{n=0}^{N} E_n)).$$

Rewrite $\mu^*(A \cap (\bigcup_{n=0}^{N} E_n))$ as follows

$$\begin{aligned}
\mu^*(A \cap (\bigcup_{n=0}^{N} E_n)) =& \mu^*(A \cap (E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash (E_1 \cup E_2)) \cup \cdots \cup (E_N \backslash \bigcup_{n=1}^{N-1} E_n))) \\
=& \mu^*(A \cap (E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash (E_1 \cup E_2)) \cup \cdots \cup (E_N \backslash \bigcup_{n=1}^{N-1} E_n)) \cap (E_N \backslash \bigcup_{n=1}^{N-1} E_n)) \\
&+ \mu^*(A \cap (E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash (E_1 \cup E_2)) \cup \cdots \cup (E_N \backslash \bigcup_{n=1}^{N-1} E_n)) \cap (E_N \backslash \bigcup_{n=1}^{N-1} E_n)^{\complement}) \\
=& \mu^*(A \cap (E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash (E_1 \cup E_2)) \cup \cdots \cup (E_N \backslash \bigcup_{n=1}^{N-1} E_n)) \cap (E_{N-1} \backslash \bigcup_{n=1}^{N-2} E_n)) \\
&+ \mu^*(A \cap (E_N \backslash \bigcup_{n=1}^{N-1} E_n)) \\
&\vdots \\
=& \mu^*(A \cap E_1) + \mu^*(A \cap (E_2 \backslash E_1)) + \cdots + \mu^*(A \cap (E_N \backslash \bigcup_{n=1}^{N-1} E_n))
\end{aligned}$$

Then we have the following

$$\lim_{N \to \infty} \mu^*(A \cap (\bigcup_{n=0}^{N} E_n)) = \mu^*(A \cap E_1) + \mu^*(A \cap (E_2 \backslash E_1)) + \cdots + \mu^*(A \cap (E_N \backslash \bigcup_{n=1}^{N-1} E_n)) + \cdots \geq \mu^*(A \cap (\bigcup_{n=0}^{\infty} E_n)).$$

2) We only need to show that $E_1, E_2, \ldots, E_n, \ldots \in B$, $E_i \cap E_j = \emptyset$ for $i \neq j$. Then $\mu^*(\bigcup\limits_{n=1}^{\infty} E_n) = \sum\limits_{n=1}^{\infty} \mu^*(E_n)$ (note: $\mu^*(\bigcup\limits_{n=1}^{\infty} E_n) \leq \sum\limits_{n=1}^{\infty} \mu^*(E_n)$.) We only have to show

$$\mu^*(\bigcup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{n} \mu^*(E_n).$$

only have to show $\mu^*(\bigcup_{n=1}^N E_n) = \sum_{n=1}^N \mu^*(E_n)$. And we have

$$\mu^*(\bigcup_{n=1}^\infty E_n) \geq \mu^*(\bigcup_{n=1}^N E_n) = \sum_{n=1}^N \mu^*(E_n), \forall N.$$

5.1 Pre-measure

- A pre-measure on a Boolean algebra. B_0 is a finitely additive measure space, $\mu_0 : B_0 \to [0, \infty]$ with the property $\mu_0(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu_0(E_n)$ whenever $E_1, E_2, \ldots, E_n, \ldots$ disjoint and $\bigcup_{n=1}^\infty E_n \in B_0$.
- **Example**: $B_0 = \{\bigcup_{i=1}^N [a_i, b_i)\}$, say $[0,1) = [0, 1-\frac{1}{2}) \cup [1-\frac{1}{2}, 1-\frac{1}{3}) \cup \cdots$. Then

$$\mu_0([0,1)) = \mu_0([0, 1-\frac{1}{2})) + \mu_0([1-\frac{1}{2}, 1-\frac{1}{3})) + \cdots$$

Counterexample: If $\mu^*([a,b)) = \begin{cases} 1 & \text{if } 1 \in (a,b] \\ 0 & \text{otherwise} \end{cases}$.

- Consider $\mu^*(E) = \inf\{\sum_{n=1}^\infty \mu_0(E_n) : E \subseteq \bigcup_{n=1}^\infty E_n, E_n \in B_0\}$.
- **Theorem (Hahn-Kolomogrov theorem)**: μ^* is an outer measure which extends μ_0.

5.2 Product space

- Consider (X, B_X), (Y, B_Y), what is a natural σ-algebra on $X \times Y$. We can define $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Hence $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$ are open, then $\{U \times Y, X \times V\}$ is a σ-algebra.
- Suppose (X, B_X, μ_X) and (Y, B_Y, μ_Y), then in $(X \times Y, B_X \times B_Y)$, we have $\mu_X \times \mu_Y(U \times V) = \mu_X(U) \times \mu_Y(V)$.
- Suppose we have $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ and $\mathcal{M} \otimes \mathcal{N} = \sigma\{U \times V : U \in \mathcal{M}, V \in \mathcal{N}\}$, which is the smallest σ-algebra such that $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Provided $(X \times Y, \mathcal{M} \otimes \mathcal{N})$, let's construct $\mu \times \nu$:
  - $\mu \times \nu(U \times V) = \mu(U) \times \nu(V)$.
  - $B_0 = \{\bigcup_{i=1}^N U_i \times V_i : U_i \in \mathcal{M}, V_i \in \mathcal{N}, i = 1, 2, \ldots N\}$. Then: $\mu \times \nu$ is well-defined and it is a pre-measure, thus $\mu \times \nu$ can be extended to a measure on $\mathcal{M} \times \mathcal{N}$.
- **Example**:
  1) $([0,1], B, m) \times ([0,1], B, m)$.
     - $m \times m([0,1] \times [0, 1/2]) = \frac{1}{2}$.
     - Let $D = \{(t,t), t \in [0,1]\}$, then $m \times m(D) = \lim_{n\to\infty} m \times m(\frac{1}{n^2}) \times n = 0$.
  2) $([0,1], B, m) \times ([0,1], B, \mu)$, where $\mu(U) = \begin{cases} |U|, & \text{if } U \text{ finite} \\ \infty, & \text{otherwise} \end{cases}$.
     - $m \times \mu([0,1] \times [0,1]) = \infty$.
     - $m \times \mu([0,1] \times \{0\}) = 1$.
     - $m \times \mu(\{0\} \times [0,1]) = 0$.
     - $m \times \mu(D) = (m \times \mu)^*(D) = \infty$.
- Assume $E \in \mathcal{M} \times \mathcal{N}$, then what is $\mu \times \nu(E)$? We can borrow the idea from double integral from calculus, if μ, ν are σ-finite,

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E_y) d\nu(y).$$

- If $E = U \times V$, then $\mu \times \nu(U \times V) = \mu(U) \cdot \nu(V)$.

  Proof.

$$\mu \times \nu(U \times V) = \int_X \nu(E_x) d\mu(x) = \int_X \nu(V) \chi_{U(x)} d\mu(x) = \nu(V) \int_X \chi_{U(x)} = \nu(V) \cdot \mu(U).$$

- Now let's take a look the $m \times \mu(D) = \int m(D_x) d\mu(x) = \int 0 d\mu(x) = 0$, also we have $\int \mu(D_y) dm(y) = \int 1 dm(y) = 1$. The inconsistency is caused by that μ is not σ-finite.
- **Lemma**: Assume $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}, E_y \in \mathcal{M}, \forall x \in X, E_y \in \mathcal{N} \forall y \in Y$.

  Proof. Let $\mathcal{C} = \{E \subseteq X \times Y : E_x \in \mathcal{N} \forall x \in X, E_y \in \mathcal{M} \forall y \in Y\}$. Next, let's show that $\mathcal{C}$ is a σ algebra. - $\emptyset, X \in \mathcal{C}$. - If $E \in \mathcal{C}$, then $E^{\complement} \in \mathcal{C}$. $(E^{\complement})_x = (E_x)^{\complement}$. - If $E_1, E_2, \ldots, E_n, \ldots \in \mathcal{C}$, then $E_1 \cup E_2 \cup \cdots \cup E_n \cup \cdots \in \mathcal{C}$. Assume $U \times V \in \mathcal{C}$, $U \in \mathcal{M}, V \in \mathcal{N}$. $\forall y, f_y : x \mapsto f(x, y)$ and $\forall x, f_y : y \mapsto f(x, y)$ are measurable.
- **Lemma**: If $E \in \mathcal{M} \otimes \mathcal{N}$ (the product of σ-algebra), then $E_y \in \mathcal{M} = \{x : (x, y) \in E\}, E_x \in \mathcal{N} = \{y : (x, y) \in E\}, \forall x \in X, y \in Y$. Then $f_y^{-1}[(r, +\infty)] = [f^{-1}(r, +\infty)]_y$. Let's show that $\forall E \in \mathcal{M} \otimes \mathcal{N}$, we have

$$\int \chi_E(x, y) d\mu(x) \times \nu(y) = \int \int \chi_E(x, y) d\mu(x) d\nu(y) = \int \int \chi_E(x, y) d\nu(y) d\mu(x)$$

  which is gives us

$$\mu \times \nu(E) = \int \mu(E_y) d\nu(y) = \int \nu(E_x) d\mu(x).$$

- Assume $\mu(X), \nu(Y) < \infty$, let $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \mu \times \nu(E) = \int \mu(E_y) d\nu(y) = \int \nu(E_x) d\mu(x)\}$. Let's show that $\mathcal{C}$ is a σ-algebra contains $\mathcal{M} \otimes \mathcal{N} = \{U \times V : U \in \mathcal{M}, V \in \mathcal{N}\}$. Let the set of subsets of X (Y) be M, which is a monotone class if

  1) $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq, U_n \in M$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} U_n \in M$.
  2) $U_1 \supseteq U_2 \supseteq \cdots U_n \supseteq$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} U_n \in M$.

- **Lemma**: The monotone class generated by an algebra is the σ-algebra generated by the algebra.

  Proof. Outline:

  1) $U \times V \in \mathcal{C}, U \in \mathcal{M}, V \in \mathcal{N}$.
  2) $\mathcal{C}$ is a monotone class.
  3) If $E = U \times V$, $E_x = \begin{cases} V & \text{if } x \in U, \\ \emptyset, & \text{otherwise} \end{cases}$, then $\mu \times \nu(U \times V) = \mu(U) \times \nu(V)$, we have

$$\int \nu(E_x) d\mu(x) = \int \chi_{U(x)} \nu(E_x) d\mu(x) = \nu(V) \cdot \mu(U).$$

  4) Increasing sequence, $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq, E_n \in \mathcal{C}, n = 1, 2, \ldots$, now let's show that $E = \bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{C}$.

$$\begin{aligned} \mu \times \nu(E) &= \mu \times \nu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \mu \times \nu(E_n) \\ &= \lim_{n \to \infty} \int \mu((E_n)_y) d\nu(y) \\ &= \int \lim_{n \to \infty} \mu((E_n)_y) d\nu(y) \\ &= \int \mu(\bigcup_{n=1}^{\infty} (E_n)_y) d\nu(y) \\ &= \int \mu(E_y) d\nu(y) \end{aligned}$$

  Decreasing sequence, $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq, E_n \in \mathcal{C}, n = 1, 2, \ldots$, now let's show that $E = \bigcap\limits_{n=1}^{\infty} E_n \in \mathcal{C}$.

$$\mu \times \nu(E) = \mu \times \nu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \mu \times \nu(E_n)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int \mu((E_n)_y) d\nu(y) \\
&= \int \lim_{n\to\infty} \mu((E_n)_y) d\nu(y) \\
&= \int \mu(\bigcap_{n=1}^{\infty} (E_n)_y) d\nu(y) \\
&= \int \mu(E_y) d\nu(y)
\end{aligned}
$$

So if μ, ν are σ-finite, then

$$\int \chi_E(x,y) d\mu \times \nu = \int \chi_E(x,y) d\mu(x) \times \nu(y) = \int \sum_{n=1}^{N} c_n \chi_{E_n}(x,y) d\mu(x) \times \nu(y) = \int\int \sum_{n=1}^{N} \chi_{E_n}(x,y) d\mu(x) d\nu(y).$$

By monotonic convergence theorem, we have

$$\int f d\mu(x) \times \nu(y) = \int\int f d\mu(x) d\nu(y) = \int\int f d\nu(y) d\mu(x),$$

$\forall f \in L^+(\mathcal{M} \times \mathcal{N})$.

- Take $f \in L^1(\mathcal{M} \times \mathcal{N})$, write $f = f_+ - f_- : f_+ < \infty, f_- < \infty$, then

$$
\begin{aligned}
\int f d\mu(x) \times \nu(y) &= \int f_+ - f_- d\mu(x) \times \nu(y) \\
&= \int f_+ d\mu(x) \times \nu(y) - \int f_- d\mu(x) \times \nu(y) \\
&= \int\int f_+ d\mu(x) d\nu(y) - \int\int f_- d\mu(x) d\nu(y) \\
&= \int\int f_+ - f_- d\mu(x) d\nu(y).
\end{aligned}
$$

- **Example**: $(\mathbb{N}, 2^{\mathbb{N}}, \mu) \times (\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is counting measure. Then

$$f(m,n) = \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } m-1 = n \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\int f(m,n) d\mu(m) \times \mu(n) = \int\int f(m,n) d\mu(m) d\mu(n) = \sum_{n=1}^{\infty} \sum_{m=1} \infty a_{m,n} = 0.$$

However,

$$\int f(m,n) d\mu(n) \times \mu(m) = \int\int f(m,n) d\mu(n) d\mu(m) = \sum_{m=1} \infty \sum_{n=1}^{\infty} a_{m,n} = 1.$$