MATH 3341: Introduction to Scientific Computing Lab

Libao Jin

University of Wyoming

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Lab 09: Ill-Conditioned Matrices and Finite Precision Arithmetic

Ill-Conditioned Matrices

Condition Number

The *condition number* of nonsingular matrix A relative to the norm $\|\cdot\|$ is

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|,$$

where the norm that is usually used is the 1-norm for matrices:

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

If the condition number is high, then the matrix is said to be *ill-conditioned*. If $\kappa(A) = \infty$, then the matrix A is singular, i.e., the matrix is not invertible.

`cond`: condition number with respect to inversion

- `cond(X)`: returns the 2-norm condition number (the ratio of the largest singular value of `X` to the smallest). Large condition numbers indicate a nearly singular matrix.
- `cond(X, P)`: returns the condition number of `X` in `P`-norm. `P` can be `1`, `2`, `inf`, or `fro`.

Hilbert Matrix

A notable example of a poorly conditioned matrix is the Hilbert matrix. A Hilbert matrix is a square matrix with elements defined by

$$H_{ij} = \frac{1}{i+j-1}.$$

For example, a 3×3 Hilbert matrix is

$$H_{3\times 3} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

Note that this matrix is symmetric and positive definite.

`hilb`: Hilbert matrix and `invhilb`: inverse Hilbert matrix

- `hilb(N)` is the `N` by `N` matrix with elements `1/(i+j-1)`, which is a famous example of a badly conditioned matrix.
- `invhilb(N)` is the inverse of the Hilbert matrix. The result is exact for `N` less than about 15.

Finite Precision Arithmetic

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Computers can only store values up to a certain level of accuracy. Past this level, the computer will round values, thus causes the round-off error. What this means is that arithmetic does not work exactly as we expect. Namely, arithmetic is no longer commutative, associative, or distributive. The lab exercises will demonstrate some of the issues that arise.

IEEE 754

https://babbage.cs.qc.cuny.edu/IEEE-754/