


MATH 3341: Introduction to Scientific Computing Lab

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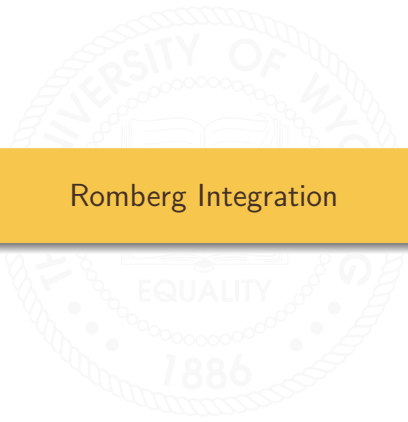
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The background of the slide features a large, faint watermark of the University of Wyoming seal. The seal is circular with a rope-like border. Inside the border, the words "UNIVERSITY OF WY" are at the top, "EQUALITY" is in the center, and "1886" is at the bottom. There are also some decorative dots and a small figure in the center.

Lab 12: Romberg Integration



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Romberg Integration



Composite Trapezoidal Rule

- Composite Trapezoidal rule for approximating the integral of a function $f(x)$ on an interval $[a, b]$ using m subintervals:

$$I = \int_a^b f(x) dx \approx \frac{1}{2} \sum_{j=0}^{m-1} (x_{j+1} - x_j)(y_{j+1} + y_j),$$

where $a = x_0 < x_1 < \cdots < x_m = b$, $y_j = f(x_j)$, $j = 0, \dots, m$.

- Let $x_{j+1} - x_j = h = (b - a)/m$, $j = 0, 1, \dots, m - 1$.

$$I \approx \frac{1}{2} \sum_{j=0}^{m-1} h[f(x_{j+1}) + f(x_j)] = \frac{h}{2} \left[f(a) + f(b) + 2 \sum_{j=1}^{m-1} f(x_j) \right],$$

where $x_j = x_0 + jh = a + jh$ for each $j = 0, 1, \dots, m$.



Composite Trapezoidal Rule

- Composite Trapezoidal Rule using m subintervals:

$$I \approx \frac{h}{2} \left[f(a) + f(b) + 2 \sum_{j=1}^{m-1} f(a + jh) \right].$$

- Finding approximation for

$m_1 = 1, m_2 = 2, m_3 = 4, \dots, m_n = 2^{n-1}$ for $n \in \mathbb{N}$. The corresponding step size h_k for each m_k is then given by $h_k = (b - a)/m_k = (b - a)/2^{k-1}$. The composite trapezoidal rule then becomes

$$I = \int_a^b f(x) dx \approx \frac{h_k}{2} \left[f(a) + f(b) + 2 \sum_{j=1}^{2^{k-1}-1} f(a + jh_k) \right].$$



Here we'll use the notation $R_{k,1}$ to denote the portion used for the trapezoidal approximation. In other words,

$$R_{1,1} = \frac{h_1}{2} [f(a) + f(b)] = \frac{b-a}{2} [f(a) + f(b)],$$

$$\begin{aligned} R_{2,1} &= \frac{h_2}{2} [f(a) + f(b) + 2f(a + h_2)] \\ &= \frac{1}{2} \frac{h_1}{2} [f(a) + f(b) + 2f(a + h_2)] \\ &= \frac{1}{2} \left\{ \frac{h_1}{2} [f(a) + f(b)] + 2 \frac{h_1}{2} f(a + h_2) \right\} \\ &= \frac{1}{2} [R_{1,1} + h_1 f(a + h_2)], \\ R_{3,1} &= \frac{1}{2} \{ R_{2,1} + h_2 [f(a + h_3) + f(a + 3h_3)] \}. \end{aligned}$$



This leads to the Trapezoidal rule in the general form

$$R_{k,1} = \frac{1}{2} \left[R_{k-1,1} + h_{k-1} \sum_{j=1}^{2^{k-2}} f(a + (2j-1)h_k) \right] \quad \text{for } k = 2, 3, \dots, n.$$

This method converges very slowly on its own. A technique called Richardson's Extrapolation is applied to speed convergence. Essentially, this performs a method of averaging previously calculated entries to obtain the next entry in the table. This is given in general form

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}.$$



This method will give us the following entries of R in a tabular format. The number of rows is determined by the value that we desire.

$$\begin{bmatrix} R_{1,1} & & & & \\ R_{2,1} & R_{2,2} & & & \\ R_{3,1} & R_{3,2} & R_{3,3} & & \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ R_{n,1} & R_{n,2} & R_{n,3} & R_{n,4} & \cdots & R_{n,n} \end{bmatrix}$$



Algorithm

Algorithm 1: Romberg Integration

Function romberg(f, a, b, n):

$h \leftarrow b - a$;

$R_{1,1} \leftarrow [f(a) + f(b)] \cdot h/2$;

for $k \leftarrow 2$ **to** n **do**

$R_{k,1} \leftarrow \frac{1}{2} \left[R_{k-1,1} + h \sum_{j=1}^{2^{k-2}} f(a + (2j-1) \cdot h/2) \right]$;

for $j \leftarrow 2$ **to** k **do**

$R_{k,j} \leftarrow R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$;

end

$h \leftarrow h/2$;

end

return $[R_{1,1}, R_{2,2}, R_{3,3}, \dots, R_{n,n}]$;

end

