MATH 5605 - Algebraic Topology Lecture Notes

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Disclaimer: there might be some typo, use this notes with caution. Also please feel free to let me know if there is anything wrong.

1 Euler's Formula

Let v be the number of vertices, e be the number of edges, f be the number of faces of a polyhedra, then we have

$$(-1)^0 v + (-1)^1 e + (-1)^2 f = v - e + f = 2.$$

2 Point Set Topology (Ch. 1-3 & Ch. 4)

Definition 2.1. An abstract topological space is a set X together with a collection of subsets of X, denoted by $\mathcal{T}$,

(a) $\emptyset, X \in \mathcal{T}$.

(b) If $u_\lambda \in \mathcal{T}, \lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} u_\lambda \in \mathcal{T}$

(c) If $u_1, \ldots, u_n \in \mathcal{T}$, then $\bigcap_{i=1}^n u_i \in \mathcal{T}$.

Any elements in $\mathcal{T}$ is called an open set.

Example 2.1.

(a) Given X, we have $\mathcal{T} = \{\emptyset, X\}$.

(b) Given X, we have $\mathcal{T} = 2^X$.

(c) Given $X = \mathbb{R}$, $\mathcal{T} = \{\text{open set in } \mathbb{R}\}$. Recall that $U \subset X$ is open, where X is a metric space if $\forall x \in U$, there is $r > 0$ such that $\mathcal{B}_r(x) \subset U$.

(d) Let X be a set, then $\mathcal{T} = \{S \subset X : S^\complement \text{ is finite or } X\}$. $S_\lambda \in \mathcal{T}, \lambda \in \Lambda$, $(\bigcup S_\lambda)^\complement = \bigcap S_\lambda^\complement \in \mathcal{T}$.

Definition 2.2. A base of a topology is a collection B of open sets such that any open set (in $\mathcal{T}$) is a union of open sets in B.

Example 2.2.

(a) $B = \{B_r(x), x \in X, r > 0\}$ and $u = \bigcup_{x \in U, B_r(x) \subset U} B_r(x)$.

(b) $X = \mathbb{R}^2$ and $d = \sqrt{x^2 + y^2}$. $B = \{B_r(x) : x \in \mathbb{R}^2, r \in \mathbb{R}^+\}$, and $B' = \{B_r(x), x \in \mathbb{Q}^2, r \in \mathbb{Q}^+\}$.

Definition 2.3. If a topological space has countable topological base, then it is called second countable.

Definition 2.4. Let $S \subset X$. A point $x \in X$ is said to be a limit point of S if for any $u \in \mathcal{T}$ with $u \ni x$. One has

$$S \cap (U \backslash \{x\}) \neq \emptyset.$$

Example 2.3.

(a) Let $X = \mathbb{R}, d = |\cdot|$, $x \in S \subset \mathbb{R}$ if and only if $\forall \varepsilon > 0$, there is $y \in S$ such that $0 < |x - y| < \varepsilon$

(b) $(X, \mathcal{T})$, finite complement topology. Let $S \subset X$, $|S| = \infty$. Pick $x \in X$, $U \ni x \implies |U^\complement| < \infty$.

(c)

Let X, Y be topological spaces.

Definition 2.5. $f : X \to Y$ is continuous if $f^{-1}(V)$ is open for all open $V subset Y$.

Example 2.4.

(a) $X = Y = \mathbb{R}$: then $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if $\forall \varepsilon > 0, \exists \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon, \forall |x - x_0| < \delta$.

Definition 2.6. $f : X \to Y$ is a homeomorphism if

(a) f is continuous.

(b) f is $1 - 1$ and onto (so $f^{-1} : Y \to X$ does exist).

(c) f^{-1} is continuous.

Definition 2.7. Let X and Y be topological spaces. If there is a homeomorphism $f : X \to Y$, then X and Y are said to be homeomorphic.

Now we have the core question in topology: To classify topological spaces up to homeomorphism.

Example 2.5.

(a) $L \approx 1$.

(b) $\mathsf{E} \approx \mathsf{F} \approx \mathsf{T} \approx \mathsf{Y}$.

(c) $(0, 1) \approx (-\infty, \infty)$.

(d) $\{(x, y) : x^2 + y^2 < 1\} \approx \mathbb{R}^2 \approx (0, 1) \times (0, 1)$.

2.1 Compactness and Connectedness

Definition 2.8 (Compactness)**.** If any open cover of K has a finite subcover, $K \subset X$ is said to be compact.

Example 2.6.

(a) Closed and bounded subset of $\mathbb{R}^n$.

(b) X, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$.

Theorem 2.1. If $f : X \to Y$ is continuous, and $K \subset X$ is compact, then $f(K)$ is compact.

Theorem 2.2. If K is compact, then K is closed.

Corollary 2.1. Suppose $f : K \to \mathbb{R}$ is continuous, and K is compact, then f is bounded and the maximum and minimum are obtainable in K.

Definition 2.9 (Connectedness)**.** X is connected if $X = X_0 \sqcup X_1$ with X_0 and X_1 open (hence closed), where $\sqcup$ means disjoint union, then $X_0 = \emptyset, X$.

Definition 2.10. X is path connected if $\forall x, y \in X$, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$, and $f(1) = y$.

Lemma 2.1. $[0, 1]$ is connected.

Theorem 2.3. Path connectedness implies connectedness.

Proof. Suppose $X = X_0 \sqcup X_1$, where X_0 and X_1 are clopen. Consider $f^{-1}(X_0), f^{-1}(X_1) \subset [0,1]$ and $f^{-1}(X_0) \cup f^{-1}(X_1) = [0,1] = f^{-1}(X_0 \cup X_1)$ and $f^{-1}(X_0) \cap f^{-1}(X_1) = \emptyset$. Since $[0,1]$ is connected, $f^{-1}(X_0) = \emptyset$ or $[0,1]$, hence $X_0 = \emptyset$ or X. □

Example 2.7. $\{(t, \sin \frac{1}{t}), 0 < t \le 1\} \cup [0,1]$ is connected but not path connected.

Let $X_\lambda, \lambda \in \Lambda$ be a family of topological spaces. Consider

$$\prod_{\lambda \in \Lambda} X_\lambda = \left\{ f : \Lambda \to \bigsqcup_{\lambda \in \Lambda} X_\lambda, f(\lambda) \in X_\lambda \right\}.$$

For any $\lambda \in \Lambda$, consider

$$\pi_\lambda : \prod_{\lambda \in \Lambda} X_\lambda \to X_\lambda, \quad (\pi_\lambda : f \mapsto f(\lambda)).$$

The product topology on $\prod_{\lambda \in \Lambda} X_\lambda$ is the weakest topology such that $\pi_\lambda, \lambda \in \Lambda$ are continuous, i.e., $\forall g$ is a topology such that $\pi_\lambda, \lambda \in \Lambda$ are continuous, then product topology is contained in g. Consider $\pi_\lambda : \prod_{\lambda \in \Lambda} X_\lambda \to X_\lambda$ is continuous, so $\pi_\lambda^{-1}(u)$ is open for all open set $u \subset X_\lambda$. Cylinder set: $u \times \prod_{\lambda' \neq \lambda} X_{\lambda'}$. More general cylinder set $(u_{\lambda_1} \times u_{\lambda_2} \times \cdots \times u_{\lambda_n}) \times \prod_{\lambda \neq \lambda_i} X_\lambda, i = 1, 2, \ldots, n$. Note: $Y^X = \{f : X \to Y\}$.

Theorem 2.4. Let $X_\lambda, \lambda \in \Lambda$ be a family of compact sets, then $\prod_{\lambda \in \Lambda} X_\lambda$ is compact.

2.2 Hausdorff Spaces

Definition 2.11. X is a *Hausdorff space* if all distinct points in X are pairwise neighborhood-seperable.

2.3 Identification Spaces

Let X be a set, P be a partition of X, i.e., there is an equivalence relation $\sim$ on X. Then $x \sim y$ if and only if x, y are in some set of the partition. Then $X/{\sim} = \{\bar{x} : x \in X\}$, $\pi : X \to X/{\sim}, x \mapsto \bar{x}$. Let X be a topological space. The quotient topology is the weakest topology on $X/{\sim}$ such that $\pi : X \to X/{\sim}$ is continuous. Let $\mathcal{T}_\sim = \{U \subset X/{\sim}\}$ is open if $\pi^{-1}(U)$ is open in X. Then $\mathcal{T}_\sim$ satisfies

(a) $\emptyset, X/{\sim} \in \mathcal{T}_\sim$.

(b) Closed under union.

(c) Closed under finite intersections.

Theorem 2.5. Given $\pi : X \to X/{\sim}$ and $f : X/{\sim} \to Z$, then f is continuous if and only if $f \circ \pi$ is continuous.

Example 2.8.

(a) Let $X = [0,1]$, then regard $0 \sim 1$, then $X/\{0,1\}, \{x : x \neq 0, 1\} \cong O$, where O is a circle (glue 0 and 1).

(b) Let $X = \mathbb{R}$, then $X/x \sim y$, where $x \sim y \iff x - y \in \mathbb{Z}$. Then $X/x \sim y \cong O$, where O is same as above.

(c) Given a disk $D_{x,r}$ centered at x with radius r. Then $D_{x,r}/\{z : |z|= 1\}, \{z : |z|< 1\} \cong S$, where S is a sphere just like Riemann sphere. $D^n_\sim \cong S^n$.

(d) Let $X = I \times I/(s, 0) \sim (s, 1), (0, t) \sim (1, t) \cong S^1 \times S^1$. $I \times I \to S^1 \times S^1, (s, t) \mapsto (e^{2\pi is}, e^{2\pi t})$, which is the composition of $\pi : I \times I \to I \times I/\sim$ and $\tilde{f} : I \times I/\sim \to S^1 \times S^1$. Note that $\tilde{f}$ is 1-1 and onto. Since $I \times I/\sim$ is compact, and $S^1 \times S^1$ is Hausdorff, then $\tilde{f}$ is a homeomorphism.

(e) $I \times I/(s, 0) \sim (s, 1), (0, t) \sim (1, 1 - t)$: Klein bottle.

(f) $I \times I/(s, 0) \sim (1 - s, 1)$: M obius strip.

(g) $S^n = \{(x_1, x_2, \ldots, x_{n+1}) : \sum_{i=1}^{n+1}|x_i|^2= 1\}$.

(h) Projective space: $\mathbb{R}^{n+1}\backslash\{0\}/\mathbf{x} \sim t\mathbf{x}, t \neq 0$.

(i) $S^n\{x, -x\}$.

(j) $\mathbb{R}^n\backslash\{0\}\backslash/t\mathbb{R}$.

(k) $D^n/\{x, -x : x \in \partial D^n\}$

Theorem 2.6. Let X be compact, Y be a Hausdorff space, $g : X \to Y$ be continuous, 1-1, and onto, then g^{-1} is continuous.

Proof. It is enough to show for any closed $E \subset X$, $(g^{-1})^{-1}(E)$ is closed. Because X is compact, E is compact, hence $g(E)$ is compact. Note that Y is Hausdorff, then $g(E)$ is closed. □

2.4 Attaching Maps and Cell Complex

Let X, Y be topological spaces. Let $A \subset X$, $f : A \to Y$,

$$X \cup_f Y := X \sqcup Y/(a, f(a)).$$

Example 2.9.

(1) Let $X = D^n$, Y be a point $\{y\}$, $f : \partial D^n \to y$, then $X \cup_f Y = S^n$.

(2) Let $X = D^n$, Y be the unit circle centered at z, $f : \partial D^n \to \partial D^n, z \to z^2$. Then $X \cup_f Y = \mathbb{R}P^2$.

Further reading: Understanding Attaching Maps, Attaching Spaces with Maps, Adjunction Space.

2.5 Topological Groups

A topological group is a group G with a topology such that $G \times G \ni (x, y) \to xy^{-1} \in G$ is continuous.

Example 2.10.

(1) Z is a topoglogical group with discrete topology;

(2) $(\mathbb{R}, +)$;

(3) $(\mathbb{R}^\times, \times)$;

(4) $GL_n(\mathbb{R}) = \{(a_{ij}), \det(a_{ij}) \neq 0\} \subset \mathbb{R}^{n^2}$;

(5) $GL_n(\mathbb{C}) \subset \mathbb{C}^{n^2}$;

(6) $U_n(\mathbb{C}) = \{u \in GL_n(\mathbb{C}), uu^* = 1\}$, which is compact. If $n = 1$, then it is complex plane.

Question: $GL_2(\mathbb{R}) \to M \mapsto \det(M)$. So not connected.

2.6 Group actions

Definition 2.12. Let G be a topological group. Let X be a topological space. A gropu action of G on X is a continuous homomorphism $G \to \text{Homeo}(X)$, where $\text{Homeo}(X) = \{\sigma : X \to X : \text{ homeomorphism}\}$ is a topological group with the pointwise convergent topology, that is, $\sigma_n \to \sigma$ if and only if $\sigma_n(x) \to \sigma$ as $n \to \infty$, $\forall x \in X$.

$G \to g \iff \sigma_g : X \to X$, note that $\sigma_g \cdot \sigma_h = \sigma_{gh}$ if $g_n \to g$, $\sigma_{g_n}(x) \to \sigma_g(x)$.

Definition 2.13. $\forall x \in X$, a stabilizer of x is defined as follows

$$\text{stab}(x) = \{g \in G, \sigma_g(x) = x\}.$$

Note that $e \in \text{stab}(x)$.

Definition 2.14. The action is *free* if $\text{stab}(x) = e, \forall x \in X$.

Definition 2.15. The *orbit* of x is

$$\text{orb}(x) = \{\sigma_g(x) : g \in G\},$$

If the action is free, then there is a $1 - 1$ correspondence such that

$$g \leftrightarrow \sigma_g(x).$$

Example 2.11.

(a) z acts on $\mathbb{R}$ by $n \leftrightarrow x \to x + n$.

(b) $\mathbb{R}$ acts on $\mathbb{R}$ by $r \leftrightarrow x \to x + r$.

(c) For rational rotation on a 2-torus, the orbit would be a circle. However, for irrational rotation on 2-torus, the orbit is dense in $\mathbb{T}^2$.

(d) $\mathbb{Z}^2$ acts on S^n by $x \to -x$.

Question: show that the orbit of irrational rotation is dense.

2.6.1 Orbit space

By definition, we have

$$X/G := X/x \sim \sigma_g(x).$$

Example 2.12.

(a) $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$.

(b) $\mathbb{R}/\mathbb{R} \cong \cdot$, where $\cdot$ is just a point.

(c) $S^n/\mathbb{Z}_2 \cong \mathbb{R}P^n$.

Let $G \curvearrowright X$, X simply connected. Then

$$\pi_1(X/G) \approx G.$$

Further reading: Kronecker flow, Examples of unique ergodicity of algebraic flows.

2.6.2 Homotopy and Fundamental Groups

Definition 2.16. Let X be a topological space. A loop is a continuous map $\alpha : [0,1] \to X$, $\alpha(0) = \alpha(1)$.

All loops form a semigroup with $\alpha \circ \beta : [0,1] \to X$,

$$\alpha \circ \beta = \begin{cases} \beta(2t) & 0 \le t \le \frac{1}{2}, \\ \alpha(2t+1) & \frac{1}{2} \le t \le 1, \end{cases}$$

and the constant map $e(t) = pt, 0 \le t \le 1$, looks like identy.

Definition 2.17. Let X and Y be topological spaces. Let $f, g : X \to Y$ (continuous). We say f is homotopic to g, written as $f \sim g$, if there is a *continuous* map $H : X \times [0,1] \to Y$ such that

$$\begin{cases} H(x,0) = f(x), \forall x \in X, \\ H(x,1) = g(x), \forall x \in X. \end{cases}$$

Let $A \subset X, f \sim g$, A if

$$\begin{cases} H(x,0) = f(x), \forall x \in X, \\ H(x,1) = g(x), \forall x \in X. \end{cases}$$

such that $H(x,t) = f(x) = g(x), x \in A$.

Example 2.13.

(a) Let X be a topological space. Let Δ be a convex set in $\mathbb{R}^n$, $f : X \to \Delta$. Pick $x_0 \in X, f(x_0)$, $H(x,t) = (1-t)f(x) + tf(x_0) \in \Delta$. Then $H(x,0) = f(x)$, $H(x,1) = f(x_0) \implies f(x) \sim f(x_0)$.

(b) $f, g : X \to S^n \subset \mathbb{R}^{n+1}$. Assume that $f(x) \ne -g(x)$, then we can construct

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$

such that

$$\begin{cases} H(x,0) = f(x), \forall x \in X, \\ H(x,1) = g(x), \forall x \in X. \end{cases}$$

Theorem 2.7. Homotopy is an equivalent relation on the set of all maps $X \to Y$.

Proof.

(a) $f \sim f$: Let $H(x,t) = f$.

(b) $f \sim g \implies g \sim f$: If $H(x,0) = f$, $H(x,1) = g$, then consider $\mathcal{H}(x,t) = H(x,1-t)$.

(c) $f \sim g, g \sim h \implies f \sim h$: $H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2}, \\ G(x,2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$.

□

Theorem 2.8. $f \sim f', g \sim g' \implies g \circ f \sim g' \circ f'$.

2.7 Fundamental Groups

Suppose that $x_0 \in X$ is the basepoint of X, and we have

$$\pi_1(X) = \{\alpha : [0,1] \to X, \alpha(0) = \alpha(1) = x_0\}/0 \sim 1$$

Theorem 2.9. $\pi_1(X)$ is a group under the multiplication. Suppose we have two loops

$$[\alpha] \cdot [\beta] = [\alpha * \beta],$$

where

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Proof.

(a) Associativity. Let $\alpha, \beta, \gamma : [0,1] \to X$ be loops with basepoint x_0.

$$([\alpha] \cdot [\beta]) \cdot [\gamma] = [(\alpha * \beta) * \gamma] = [\alpha * (\beta * \gamma)] = [\alpha] \cdot ([\beta] \cdot [\gamma]),$$

since $(\alpha * \beta) * \gamma ?\,\text{rel}\ \overset{\sim}{\{}0,1\}\alpha * (\beta * \gamma)$.

(b) Identity. Let $e = \{e(t) = x_0 : 0 \le t \le 1\}$. Let α be an arbirary loop, then

$$[\alpha] \cdot [e] = [a * e] = [a].$$

Since $a * e \sim \alpha$, let

$$(\alpha * e)(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2}, \\ x_0 & \frac{1}{2} \le t \le 1. \end{cases}$$

(c) Inverse. Let α be a loop with basepoint x_0. Define $\alpha^{-1}(t) = \alpha(1-t), 0 \le t \le 1$. Show that

$$\alpha * \alpha^{-1} \sim e, \quad \alpha^{-1} * \alpha \sim e.$$

The homotopy

$$H(t,s) = \begin{cases} \alpha(t) & 0 \le t \le 1-s \\ \alpha(1-s) = \alpha^{-1}(s) & 1-s \le t \le 1+s \\ \alpha^{-1}(1-t) & 1+s \le t \le 2. \end{cases}$$

□

Theorem 2.10. Let X be a topological space. Let $p, q \in X$ such that p, q are connected by a path, $\pi_1(X, p) \cong \pi_1(X, q)$.

Proof. $\Phi_r : \pi_1(X, p) \ni [\alpha] \mapsto [\gamma^{-1} * \alpha * \gamma] \in \pi_1(X, q)$.

- This is well-defined: if $\alpha \sim \beta$, $\gamma^{-1} * \alpha * \gamma \sim \gamma^{-1}\beta * \gamma$ under the relation $\{0, 1\}$.
- This is a homomorphism. $\alpha * \beta \mapsto \gamma^{-1} * \alpha * \beta * \gamma$, $\alpha \mapsto \gamma^{-1}\alpha\gamma$, $\beta \mapsto \gamma^{-1}\beta\gamma$. Then under relation $\{0, 1\}$:
$$\gamma^{-1} * \alpha * \beta * \gamma \sim \gamma^{-1} * \alpha * \gamma * \gamma^{-1}\beta * \gamma.$$
- $\Phi_{\gamma^{-1}}$ is the inverse of Φ_γ.

□

Theorem 2.11. Let $f : (X, x_0) \to (Y, f(x_0))$ be continuous, then f induces a homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$

Proof. $f_* : \pi_1(X, x_0) \to [\alpha] \mapsto [f \circ \alpha] \in \pi_1(Y, f(x_0))$, $\alpha : (\mathbb{T}, *) \to (X, *)$, $f : (X, *) \to (Y, *)$. □

Further reading: Fundamental group, Notes on the Fundamental Group by Aaron Landesman, Functor.
$X \xrightarrow{f} Y \xrightarrow{g} z$, then $(g \circ f)_* = (g)_* \circ (f)_*$.
Assume X is homeomorphic to Y, i.e., $f : X \to Y$, $g : Y \to X$, then $f \circ g = 1_X$, and $g \circ f = 1_X$. Then

$$g_* \circ f_* = (g \circ f)_* = (1_X)_* = \mathrm{id}_{\pi_1(X)}$$
$$f_* \circ g_* = (f \circ g)_* = (1_Y)_* = \mathrm{id}_{\pi_1(Y)}$$

That implies
$$\pi_1(X) \cong \pi_1(Y).$$

Example 2.14.

(a) $X = \{x_0\}$, then the fundamental group is $\pi_1(X) = \{e\}$.

(b) $X = D = \{z : |z| \leq 1\}$. Let $\alpha : [0, 1] \to D$, where $\alpha(0) = \alpha(1) = 0$. Consider $H(s, t) = (1 - t)\alpha(s) + t \cdot 0$. $\alpha ? \operatorname{rel} \overset{\sim}{\{0,1\}} 0$. In this case, $\pi_1(X) = \{e\}$.

(c) $X = \mathbb{R}, \pi_1(\mathbb{R}, 0) = \{e\}$.

(d) $X = S^1$, there are obvious loops: $t \to e^{2\pi it}$.

Let $\alpha : [0, 1] \to S^1$ be a path with $\alpha(0) = 1$. Question: Does there exist $\tilde{\alpha} : [0, 1] \to \mathbb{R}$, $\tilde{\alpha}(0) = 0$? Yes. Let $H(t, s) = (1 - s)\tilde{\alpha}(t) + snt$ such that

$$H(t, 0) = \tilde{\alpha}(t), \quad H(t, 1) = nt.H(0, s) = (1 - s)0 + 0 = 0, \quad H(1, s) = (1 - s)n + sn = n.$$

Further reading: Homotopy lifting property, Covering space, Path lifting and the fundamental group, Lecture XVI - Lifting of paths and homotopies.

Theorem 2.12 (Path lifting theorem)**.** Let $\alpha : [0,1] \to S^1$ be a loop with base point x_0. There is a unique continuous map $\tilde{\alpha} : [0,1] \to \mathbb{R}^1$,

(a) $\alpha(0) = 0$.

(b) $\pi \circ \tilde{\alpha} = \alpha$.

Proof. Write $S^1 = S^+ \cup S^-$, where S^+ is the sphere with south pole removed and S^- is the sphere with north pole removed. Then if the image of α is contained S^+ (or S^-), then α has a lifting. In general, we can map the interval $[0,1]$ to S^1. Then there is $\delta > 0$ such that $\alpha([t, t+\delta])$ is either in S^+ or S^-. □

Let α be a loop, it can be lifted to $\tilde{\alpha} : [0,1] \to \mathbb{R}^1, \tilde{\alpha}(0) = 0, \tilde{\alpha}(1) = n$. Then the homotopy:

$$H(t,s) = (1-s)\tilde{\alpha}(t) + snt \implies \tilde{\alpha}(t) \sim nt.$$

Consider $\alpha : t \to e^{2\pi i n t}, n \neq 0$. Assume that $\alpha \sim (\mapsto 1)$, i.e., $\exists H : [0,1] \times [0,1] \to S^1$ such that

$$H(t,0) = e^{2\pi i n t}, H(t,1) = 1, H(0,s) = 1, H(1,s) = 1.$$

Corollary 2.2. $\pi_1(S^1) \cong \mathbb{Z}$.

Further reading: Lifts of paths,

Example 2.15.

(a) $\pi_1(\mathbb{R}^1) = \{0\}$, simply connected.

(b) $\forall x \in S^1$, there is $U \ni x$ such that

$$\pi^{-1}(U) = \bigsqcup_{n=-\infty}^{\infty} V_n,$$

and $\pi|_{V_n}$ is a homeomorphism. Consider a simply connnected space X. Let $G \curvearrowright X$ be a group action. Assume for any $x \in X$, there exists $U \ni x$,

$$g(U) \cap U = \emptyset, \quad g \notin e.$$

Then

$$\pi_1(X/G) \cong G.$$

Theorem 2.13. $\pi_1(S^n) = \{0\}, n \geq 2$.

Proof. Consider $\mathbb{Z}^2$ acting on S^n by $x \mapsto -x, x \in S^n$. Let $\alpha : [0,1] \to S^n$ be a based at x_0. Then there is a partition. □

Let X be a simply connected space $\pi_1(X) = \{0\}$. Let G acting on X such that $\forall x \in X$, there is $U \ni x$ such that $g(U) \cap U = \emptyset, \forall g \neq e$. Then $\pi_1(X/G) \cong G$.

Example 2.16.

(a) $X = \mathbb{R}$, $G = \mathbb{Z}$, $n(x) = n + x$. Then $S^1 \cong \mathbb{R}/\mathbb{Z}$, $\pi_1(S^1) = \mathbb{Z}$.

(b) $X = S^n, n \geq 2$, $G = \mathbb{Z}/2\mathbb{Z}$ acting on S^n, $x \mapsto -x$.

$$\mathbb{R}P^n = S^n/\{x, -x\} = X/G, \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}.$$

(c) $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$, $k(x) = k + x, k \in \mathbb{Z}^n, x \in \mathbb{R}^n$. $\mathbb{R}^n/\mathbb{Z}^n = X/G = S^1 \times S^1$, $\pi_1((S^1)^n) = \mathbb{Z}^n$.

Question: What is $\pi_1(\mathbb{R}P^n \times S^1)$? Answer:

$$\pi_1(\mathbb{R}P^n \times S^1) = \pi_1(\mathbb{R}P^n) \times \pi_1(S^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.$$

Theorem 2.14. $\pi_1(X \times Y) = \pi_1(X) \times \pi_2(Y)$.

Proof. Let $P_1 : X \times Y \to X$, $P_2 : X \times Y \to Y$, be projections. Pick $\alpha : [0,1] \to X \times Y$, $\alpha(0) = \alpha(1) = (x_0, y_0)$. Consider $P_1 \circ \alpha$ the loop in X with base point x_0 and $P_2 \circ \alpha$ the loop in X with base point y_0. If $\alpha \overset{H}{\sim} \beta$ with relation $0, 1$, where $H : [0,1] \times [0,1] \to X \times Y$, $P_1 \circ \alpha \overset{P_1 \circ H}{\sim} P_1 \circ \beta$, $P_2 \circ \alpha \overset{P_2 \circ H}{\sim} P_2 \circ \beta$, then

$$\pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y), P : [\alpha] \mapsto ([P_1 \circ \alpha], [P_2 \circ \alpha]).$$

Need to show P is $1-1$ and onto. To show P is one-to-one: $\langle\alpha\rangle \in \pi_1(X \times Y)$ if $P(\langle\alpha\rangle) = \{e\}$. $P_1\alpha \overset{H_1}{\sim} x_0$ with relation $\{0,1\}$, where $H_1 : [0,1] \times [0,1] \to X$. $P_2\alpha \overset{H_2}{\sim} y_0$ with relation $\{0,1\}$, where $H_1 : [0,1] \times [0,1] \to Y$. Now $H := H_1 \times H_2$. □

Theorem 2.15 (Brouwer fixed-point theorem)**.** Let D^n be the n-dimensional disk. Then, for any conitnuous map $f : D^n \to D^n$, there is $x^* \in D^n$ such that $f(x^*) = x^*$.

Proof. Consider $n = 2$. Suppose $f : D^2 \to D^2$ has no fixed point $f(x) \neq x, \forall x \in D$. Then there is a continuous map $\partial D \overset{g}{\to} D \overset{h}{\to} \partial D \cong S^1, x \mapsto x \mapsto t(x)$. Then $g \circ h = \mathrm{id}_{\partial D}$. Now consider n. Suppose there is no fixed points, then consider then map $P : D^n \to S^{n-1}(\cong \partial D^n), x \mapsto t(x)$. $S^{n-1} \overset{\iota}{\curvearrowright} D^n \overset{P}{\to} S^{n-1}$, where

$$P \circ \iota = \mathrm{id}_{S^{n-1}}, P_* \circ \iota_* = \mathrm{id}_* \text{ on } \pi_1(S^{n-1}) = \mathbb{Z}.$$

if and only if

$$\pi_1(S^{n-1}) \overset{\iota_*}{\to} \pi_1(D^n) = \{0\} \overset{P_*}{\to} \pi_1(S^{n-1}).$$

Contradiction. □

Further reading: Group action,

Example 2.17.

(a) $\pi_1(S^1) = \mathbb{Z}$.

(b) $\pi_1(D^n) = \{e\}$.

(c) $\pi_1(X/G) = G$.

(d) $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

(e) $\pi_1(S^n) = \{e\}, n \geq 2$.

(f) $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$.

(g) $\pi_1(\mathbb{R}^2 \backslash \{0,0\}) \sim \pi_1(S^1) = \mathbb{Z}$.

(h) $\pi_1(CX) \sim \pi_1(\{x_0\}) = \{x_0\}$.

Definition 2.18. Let X, Y be topological spaces, we say that X and Y have the same homotopy type, or, X is homotopic to Y if there exists $X \Leftrightarrow Y$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.

Example 2.18.

(a) If X and Y are homeomorphic, then $X \sim Y$.

(b) $\mathbb{R}^n \backslash \{0\} \sim S^{n-1}$, where $f : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$, $x \mapsto \frac{x}{\|x\|}$. and g is the embedding map. Therefore,

$$f \circ g(x) = x, \forall x \in S^{n-1}$$
$$g \circ f(x) = \frac{x}{\|x\|}, \forall x \in \mathbb{R}^n \backslash \{0\}.$$

(c) $S^1 \times [0,1]/(S^1 \times \{1\}) = CX$. is a cone.

Theorem 2.16. Let X, Y be path connected spaces, if $X \sim Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof. Consider

$$\pi_1(g) \circ \pi_1(f) = \pi_1(g \circ f),$$
$$\pi_1(f) \circ \pi_1(g) = \pi_1(f \circ g).$$

Then by Lemma 2.2, we have

$$\pi_1(g) \circ \pi_1(f) = \pi_1(g \circ f) = (\gamma_1)_*,$$
$$\pi_1(f) \circ \pi_1(g) = \pi_1(f \circ g) = (\gamma_2)_*,$$

where $(\gamma_1)_* : \pi_1(X, p) \to \pi_1(X, fg(p))$, and $(\gamma_2)_* : \pi_1(X, p) \to \pi_1(X, gf(p))$. □

Exercise 2.1. $S^1 \times S^1 \backslash \{x_0\}$.

Lemma 2.2. If $f \overset{H}{\sim} g : X \to Y$. Then $\pi_1(f) : \pi_1(X,p) \to \pi_{Y,f(p)}$ and $\pi_1(g) : \pi_1(X,p) \to \pi_1(Y, g(p))$. Then

$$\pi_1(X,p) \overset{\pi_1(g)}{\to} \pi_1(Y, g(p)) : \pi_1(X,p) \overset{\pi_1(f)}{\to} \pi_1(Y, f(p)) \overset{\gamma}{\to} \pi_1(Y, g(p)).$$

Proof. Let α be a loop in X with basepoint p. Then $H(x,t) \circ \alpha = H(\alpha(s), t) : f \circ \alpha \to g \circ \alpha$. □

Further reading: Poincaré Conjecture.

2.8 Covering Spaces

Definition 2.19. A map $P : E \to X$ is a covering map and E is said to be a covering space of X if for any x $x \in X$, there is an open neighborhood $U \ni x$ such that $P^{-1}(U)$ is the disjoint union of open sets $S_i \subset E$, each S_i is mapped homeomoprhically by P onto U. Each S_i is called a sheet of U.

Example 2.19.

(a) $\mathbb{R} \overset{e^{2\pi i t}}{\to} \mathbb{T}$.

(b) $X \overset{id}{\to} X$.

(c) $S^n \to \mathbb{R}P^n, x \mapsto \{x, -x\}$.

(d) $\mathbb{T} \to \mathbb{T}, z \mapsto z^n$.

2.8.1 Unique lifting

Theorem 2.17. Assume that Y is connected, then the lifting, if it exists, then it is unique.

Proof. Let $f : (Y, y_0) \to (X, x_0)$, assume that it has two liftups, f', f''. Consider

$$A = \{y \in Y : f'(y) = f''(y)\} \ni y_0, D = \{y \in Y : f'(y) \neq f''(y)\}.$$

If A and D are open, then $A = Y$. Let $q = f'(y) = f''(y) \in S_i \subset (E, e_0)$. Then $y' \in (f')^{-1}(S_i) \cap (f'')^{-1}(S_i)$, then $f'(y'), f''(y') \in S_i$, and $Pf''(y') = Pf'(y') = f(y')$, therefore A is open. In the same flavor, D is open. □

Definition 2.20. A homeomorphism $\varphi : E \to E$ is said to be a covering transformation if $\varphi : E \to E$, $P : E \to X$, i.e., $P \circ \varphi = P$.

Remark 2.1. All the covering transofmormations form a group.

Further reading: deck transformation, covering spaces.

Lemma 2.3. If γ is a path in X which begins at p, there is a unique path $\tilde{\gamma}$ in $\tilde{X}$ begins at q and satifies $\pi \circ \tilde{\gamma} = \gamma$.

Lemma 2.4. If $F : I \times I \to X$ is map such that $F(0, t) = F(1, t) = p$ for $0 \leq t \leq 1$, there is a unique map $\tilde{F} : I \times I \to \tilde{X}$ which satisfies $\pi \circ \tilde{F} = F$ and $\tilde{F}(0, t) = q, 0 \leq t \leq 1$.

Definition 2.21. A homeomorphism $\varphi : E \to E$ is said to be deck transformation if $P \circ \varphi = P$.

Proposition 2.1. All deck transformations form a group.

Proof.

(a) id is a deck transofmration.

(b) $P \circ \varphi = P \implies P = P \circ \varphi \circ \varphi = P \circ \varphi^{-1}$.

(c) $P \circ \varphi_1 = P$ and $P \circ \varphi_2 = P$, then $P \circ \varphi_1 \circ \varphi_2 = P \circ \varphi_2 = P$.

□

Theorem 2.18. Assume that E is simply connected and locally path connected, then the group of all deck transformations is canonically isomorphic to $\pi_1(X)$.

Proof. Let G be the group of deck transformations, then we can construct a homeomorphism $\chi : G \to \pi_1(X), \varphi \mapsto \chi(\varphi)$. Pick $\varphi : (E, e_0) \to (E, \varphi(e_0))$ such that $P : (E, e_0) \to (X, x_0)$ and $P : (E, \varphi(e_0)) \to (X, x_0)$. Choose a path φ from e_0 to $\varphi(e_0)$, then $P \circ \gamma$ is a loop in X with basepoint x_0. Then

$$\chi(\varphi) := [P \circ \gamma] \in \pi_1(X, x_0).$$

Since E is simply connected, then the homotopy class of γ only depends on e_0 and $\varphi(e_0)$. Then χ is a homeomorphism, then $\chi(\psi \circ \varphi) = \chi(\psi) \cdot \chi(\varphi)$, i.e.,

$$[P \circ \gamma_1 \circ \gamma_2] = [P \circ \gamma_1] \cdot [P \circ \gamma_2]$$

□

Remark 2.2. Assume that E is connected, $P : (E, e_0) \to (X, x_0)$, $\varphi_1 : (E, e_0) \to (E, \varphi_1(e_0))$, $\varphi_2 : (E, e_0) \to (E, \varphi_2(e_0))$. Then if $\varphi_1(e_0) = \varphi_2(e_0)$, then $\varphi_1 = \varphi_2$.

2.8.2 Lifting criteria

Theorem 2.19 (Map-lifting theorem)**.** There is a lift of f which takes r to q if and only if $f_*(\pi_1(Y, r)) \subset \pi_*(H)$, and this lift is unique.

Theorem 2.20. If X is semi-locally simply connected, then X has a universal covering space $\tilde{X}$.

Definition 2.22. X is *semi-locally simply connectec* if $\forall x \in X$, there is $U \ni x$ such that any loop $\alpha : [0, 1] \to U$ is homotopically trivial in U.

Example 2.20. All manifolds.

3 Simplicial complex

Definition 3.1. A simplicial complex K consists of the set of vertices V and a set S of finite subset of V (each subset forms a simplex) such that

(a) $\{v\} \in S, \forall v \in V$.

(b) If $\{v_1, v_2, \dots, v_n\} \in S$, then any subset of $\{v_1, \dots, v_n\}$ is in S.

Example 3.1. $V = \{1, 2, 3, 4\}$.

$$S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}.$$

Assume $|v| < \infty$. Let $v = \{v_1, v_2, \dots, v_n\}$. Consider $\mathbb{R}^n$, the basis of $\mathbb{R}^n$, $\{e_1, e_2, \dots, e_n\}$. Then

$$|K| = \{\alpha : V \to [0, 1]\}$$

such that $\sum_{v\in V} \alpha(v) = 1$ and $\{v : \alpha(v) = 0\}$ is a simplex.

Definition 3.2. A triangulation of a topological space X consists of a simplicial complex K and a homeomorphism $h : |K| \to X$.

The dimension of a simplicial complex K is the largest n such that K contains a n-simplex.

3.1 Barycentric subdivision

For a given K, fix $|K|$.

Lemma 3.1.

(a) Each simplex of $K^{(1)}$ is contained in a simplex of K.

(b) $|K^{(1)}| = |K|$.

(c) If d is the dimension of K, then $\text{mesh}(K^{(1)}) \le \frac{d}{d+1} \text{mesh}(K)$, where $\text{mesh}(K) = \max_i \text{diam}(K_i)$.

Definition 3.3. Let K and L be simplicial complexes. A function $f : |K| \to |L|$ is simplicial if it takes simpliexes of K onto simplexes of L, i.e., if $\{v_0, v_1, \dots, v_d\}$ is simplex of K, then $\{f(v_0), f(v_1), \dots, f(v_d)\}$ is a simplex, and we have

$$f(t_0v_0 + \cdots + t_dv_d) = t_0f(v_0) + \cdots + t_df(v_d).$$

Definition 3.4. A simplecial map $s : |K| \to |L|$ is a simplicial approximation of a continuous map $f : |K| \to |L|$ if for any $x \in |K|$, the image of $s(x)$ is contained in the carrier of $f(x)$.

f is homotopic to s by $H(x, t) = tf(x) + (1 - t)s(x)$, because $f(x), s(x)$ are in the same simplex.

Theorem 3.1. Let $f : |K| \to |L|$ be a continous map. Then, with m large enough, there is a simplicial approximation.

$$s : |K^{(m)}| \to |L|.$$

That is, $s(x)$ is in the carrier of $f(x)$, i.e., $\forall x$, $s(x)$ and $f(x)$ are in the same simplex.

Proof. Assume for each vertex $u \in K$, there is $v \in L$ such that

$$f(\text{star}(u)) \subset \text{star}(v).$$

How do we construct $s : K \to L$? For each vertex u of K, pick v, a vertex of L such that

$$f(\text{star}(u)) \subset \text{star}(v).$$

Then $s : u \mapsto v$. To see that s induces a simplicial map. Need to show that if $u_1, u_2, \ldots, u_n$ are in one simplex, then $v_1, v_2, \ldots, v_n$ are in the one simplex. That implies

$$f(\bigcap_{i=1}^{n} \text{star}(u_i)) \subset \bigcap_{i=1}^{n} f(\text{star}(u_i)) \subset \bigcap_{i=1}^{n} \text{star}(v_i).$$

Consider the open cover of L: $\{\text{star}(v), v \in L\}$. Then there is a $\delta > 0$ such that if for any $x \in |K|$, $f(B_\delta(x)) \subset \text{star}(v)$ for some v. Then choose m sufficiently large such that

$$|\text{star}(u)| < \delta, u \in K^{(m)}.$$

□

Remark 3.1. $f \sim s$.

Consider a simplex K, fix $|K|$. Let v be a vertex of K. Then the union of the interior of simplexes contan v as a vertex is called $\text{star}(v)$, which is an open neighborhood of v.

Lemma 3.2. Vertices $v_1, v_2, \ldots, v_m$ span a simplex if and only if

$$\bigcap_{i=1}^{n} \text{star}(v_i) \neq \emptyset.$$

3.2 Edge group

K simplicial complex, v vertex, consider edge of K, $v_1, v_2, \ldots, v_n$,

(a) $v_i, i = 1, \ldots, n$ are vertices of K.

(b) $v_i v_{i+1}$ is a simplex.

(c) $v_1 = v_n = v$.

The equivalence relation generated by the span, we have if $v_{n_1}, v_{n_2}, \ldots, v_{n_k}$ span a simplex, then $v_{n_1}, v_{n_2}, \ldots, v_{n_k} \sim v_{n_1} \sim v_{n_k}$.

Definition 3.5. $E(k,v) := \{\text{edge loop}\}/\sim$.

Theorem 3.2. $E(k,v) \cong \pi_1(|K|, v)$.

Let K be a simplicial complex, L be a subcomplex such that L contains all vertices of K, and $|L|$ simply connected (connected). Let $G(K,L)$ be the group generated by g_{v_i,v_j}, where v_i, v_j are vertices of K, which span a simplex of K with respect to the following relation:

$$\begin{cases} g_{v_i,v_j} = e & \text{if } L_{0,1} = \operatorname{span}\{v_i, v_j\}, \\ g_{v_i,v_j} g_{v_j,v_k} = g_{v_i,v_k} & \text{if } K_{0,1,2} = \operatorname{span}\{v_i, v_j, v_k\}, \end{cases}$$

where $L_{0,1}$ is the collection of 0-simplex and 1-simplex of L, and $K_{0,1,2}$ is the collection of 0, 1, 2-simplex of K. Then

$$G(K,K) = F_{\{g_{v_i,v_j}\}} / \{g_{v_i,v_j}, v_i v_j \in L, g_{v_i v_j} g_{v_j v_j} g^{-1}_{v_i v_j}\}$$

Theorem 3.3. $G(K,L) \cong E(K,v) \cong \pi_1(|K|, v)$.

Proof. Let $\varphi : G(K,L) \to E(K,v)$, $\psi : E(K,v) \to G(K,L)$. For each v_i, choose a path E_i from v to v_i. Let v_1, v_2 be vertices of K such that $v_1 v_2$ span a simplex. Then we can define φ by generator $g_{v_1 v_2}$ as $\varphi : g_{v_1 v_2} \mapsto E_1 v_1 v_2 E_2^{-1}$. To see that φ induces a homomorphism, need to show that φ preserve the relations.

(a) If v_{i_1}, v_{i_2} span a simplex in L. Then $g_{v_{i_1}, v_{i_2}} \mapsto E_{i_1} v_{i_1} v_{i_2} E_{i_2}^{-1} = e$.

(b) Let $g_{v_i v_j} \mapsto E_i v_i v_j E_j^{-1}$,$g_{v_j v_k} \mapsto E_j v_j v_k E_k^{-1}$, $g_{v_i v_k} \mapsto E_i v_i v_k E_k^{-1}$. Then

$$\begin{aligned} g_{v_i v_j} g_{v_j v_k} \mapsto (E_i v_i v_j E_j^{-1})(E_j v_j v_k E_k^{-1}) &\iff g_{v_i v_j} g_{v_j v_k} \mapsto E_i v_i v_j (E_j^{-1} E_j) v_j v_k E_k^{-1} \\ &\iff g_{v_i v_j} g_{v_j v_k} \mapsto E_i v_i v_j v_j v_k E_k^{-1} \\ &\iff g_{v_i v_j} g_{v_j v_k} \mapsto E_i v_i v_j v_k E_k^{-1} \\ &\iff g_{v_i v_j} g_{v_j v_k} \mapsto E_i v_i v_k E_k^{-1} \end{aligned}$$

□

Example 3.2.

(a) 1-sphere, $G(K,L) = \langle g_1 \rangle = \mathbb{Z}$.

(b) Figure 8. $G(K,L) = \langle g_1, h_1 \rangle = \mathbb{Z} * \mathbb{Z}$, where $*$ is free product.

(c) Let K be connected, 1-connected simplicial complex, L is the spanning tree, and $G(K,L) = \langle e \rangle$, where e are edges not in L.

Consider $K_1, K_2 \subset K$, K_1, K_2, K are simplicial complexes such that $K_1 \cup K_2 = K$, and $|K_1 \cap K_2|$ connected. *Question*: Can we compute $\pi_1(K_1 \cup K_2)$ in terms of $\pi_1(K_1)$, $\pi_1(K_2)$, and $\pi_1(K_1 \cap K_2)$.

Theorem 3.4 (Van Kampen's Theorem)**.** The fundamental gropu of $|J \cup K|$ based at v is obtained from the free product $\pi_1(|J|, v) * \pi_1(|K|, v)$ by adding the relations $j_*(z) = k_*(z)$ for all $z \in \pi_1(|J \cap K|, v)$.

In the topological sense:

Theorem 3.5 (Van Kampen's Theorem)**.** The fundamental gropu of $X_1 \cup X_2$ based at v is obtained from the free product $\pi_1(X_1) * \pi_1(X_2)$ by adding the relations $j_*(z) = k_*(z)$ for all $z \in \pi_1(X_1 \cap X_2)$.

Example 3.3.

(a) Given $\mathbb{T}^2 = |K_1| \cup |K_2|$, where $K_2 = D$ is a disk, and $K_1 = \mathbb{T}^2 \backslash K_1$. It is known that $\pi_1(|K_1|) = \langle a, b \rangle$, $\pi_1(|K_2|) = 0$, and $|K_1| \cap |K_2| = S^1$ whose fundamental group is $\pi_1(|K_1| \cap |K_2|) = \mathbb{Z}$. Then

$$\pi_1(\mathbb{T}^2) = \mathbb{F}^2 * \{e\}/(ab^{-1}a^{-1}b = e) = \mathbb{Z}^2 = \{a, b | aba^{-1}b^{-1} = e\}.$$

(b) $\mathbb{R}P^2$.

Further reading: free group

Theorem 3.6. Let G be a subgroup of $F_2 = \langle a, b \rangle$, then $G \cong \langle g_1, g_2, \ldots, g_n \rangle$ for some $n \in \mathbb{N} \cup \{+\infty\}$.

Proof. Outline: Identify G as the fundamental group of some "infinite" 1-dimnesional simplicial complex. □

Further reading: Cayley graph.

3.3 Simplicial Homology

Let K be a simplicial complex, then $H_0(|K|), H_1(|K|), \cdots$, where $H_n(|K|)$ (abelian) is the n-th homology of $|K|$.

3.3.1 Cycle and Boundaries

Let $C_q(k)$ is a free abelian group generated by q-simplexes of K,

(a) $c \in C_q(k), c = m_1\Delta_1 + \cdots + m_n\Delta_n$, where $\Delta_1, \Delta_2, \ldots, \Delta_n$ are q-simplexes $m_i \in \mathbb{Z}, i = 1, 2, \ldots, n$, is a cycle of

$$\partial C = 0,$$

where

$$\partial([v_0 v_1 \cdots v_q]) = \sum_{i=0}^{q} (-1)^i [v_0 \cdots \hat{v}_i \cdots v_q].$$

(b) $c \in C_q(K)$ is a boundary of $c = \partial(c_1)$ for some $c_1 \in c_{\varepsilon+1}(K)$.

Example 3.4. Given a 2-simplex, $v_1 v_2 v_3 v_4$, then

$$[v_1 v_2] + [v_2 v_3] + [v_3 v_4] + [v_4 v_1] = \partial([v_1 v_2 v_3] + [v_1 v_3 v_4]).$$

Lemma 3.3. A boundary is a cycle, i.e.,

$$\operatorname{Im} \partial \subset \ker \partial \iff \partial \cdot \partial = 0.$$

Proof. It suffices to show that $\partial(\partial(c)) = 0$, where $c = [v_0 v_1 \cdots v_q]$. Then

$$\begin{aligned}
\partial(\partial c) &= \partial(\partial[v_0 v_1 \cdots v_q]) \\
&= \partial(\sum_{i=1}^{q}(-1)^i [v_0 \cdots \hat{v}_i \cdots v_q]) \\
&= \sum_{i=1}^{q}(-1)^i \partial([v_0 \cdots \hat{v}_i \cdots v_q]) \\
&= \sum_{i=1}^{q}(-1)^i \left(\sum_{j=0}^{i-1}(-1)^j \partial([v_0 \cdots \hat{v}_i \cdots v_q]) + \sum_{j=i+1}^{n}(-1)^{j+1}\partial([v_0 \cdots \hat{v}_i \cdots v_q])\right)
\end{aligned}$$

□

Theorem 3.7. $H_1(K) = \pi_1(|K|)/[\pi_1(|K|), \pi_1(|K|)]$.

Proof. Construct an onto map: $\varphi : \pi_1(|K|) \to H_1(K)$ such that $\ker \varphi = [\pi_1(|K|), \pi_1(|K|)]$. Identify $\pi_1(|K|) = E(K, V)$. Let $\alpha = E(K, V)$, pick an edge loop, $v_1 v_2 \cdots v_n$, where $v_1 = v_n$. Let

$$\alpha' = [v_1, v_2] + [v_2, v_3] + \cdots + [v_{n-1}, v_n] \in C_1.$$

Is α' a cycle? Yes.

$$\begin{aligned}
\partial(\alpha') &= \partial([v_1, v_2] + [v_2, v_3] + \cdots + [v_{n-1}, v_n]) \\
&= \partial([v_1, v_2]) + \partial([v_2, v_3]) + \cdots + \partial([v_{n-1}, v_n]) \\
&= v_1 - v_2 + v_2 - v_3 + \cdots + v_{n-1} - v_n \\
&= v_1 - v_n \\
&= 0.
\end{aligned}$$

To show this map is well-defined, we need to have

$$\begin{aligned}
\alpha &= [v_1, v_2] + [v_2, v_3] + \cdots + [v_i, v_j] + [v_j, v_k] + \cdots + [v_{n-1}, v_n], \\
\beta &= [v_1, v_2] + [v_2, v_3] + \cdots + [v_i, v_k] + \cdots + [v_{n-1}, v_n], \\
\alpha - \beta &= \partial(v_i, v_j, v_k).
\end{aligned}$$

□

What is the kernel of φ? Let $\alpha = v_0 v_1 \cdots v_n$ be an edge group such that

$$[v_0, v_1] + \cdots + [v_{n-1}, v_n] = \partial(\Delta_1 + \cdots + \Delta_k),$$

where $\Delta_i = [a_i, b_i, c_i], i = 1, 2, \ldots, k$ are 2-simplexes of K. Consider $\gamma_i a_i c_i b_i a_i \gamma_i^{-1}$ for Δ_i. Consider

$$\beta = \prod_{i=1}^{k} \gamma_i a_i c_i b_i a_i \gamma_i^{-1}.$$

Note that $\beta \sim_h V$ in $\pi_1(|K|)$, where $[\alpha \cdot \beta] = [\alpha]$. Consider $\alpha \cdot \beta$ as an element in $C_1(K)$. Then

$$\begin{aligned} \alpha &\sim \partial(\Delta_1 + \Delta_2 + \cdots + \Delta_k) \\ &= \partial(\Delta_1) + \partial(\Delta_2) + \cdots + \partial(\Delta_k) \\ &= \sum_{i=1}^{k}([b_i, c_i] + [c_i, a_i] + [a_i, b_i]), \\ \beta &= \sum_{i=1}^{k}([c_i, b_i] + [a_i, c_i] + [b_i, a_i]), \end{aligned}$$

Therefore $\alpha \cdot \beta = 0$. If a 1-edge appears in $\alpha \cdot \beta$, it appears twice, one is positive and the other is negative. Consider

$$E(K, v) \xrightarrow{\theta} G(K, L) \xrightarrow{\pi} GL(K, L)/[G, G].$$

Recall that $G(K, L)$ is the group generated by $g_{[v_i,v_j]}$ with relations: 1. $g_{[v_i,v_j]} = 1$ if $[v_i, v_j] \in L$; $g_{[v_i,v_j]}g_{[v_j,v_k]} = g_{[v_i,v_k]}$. Therefore,

$$[\alpha \cdot \beta] \mapsto \theta([\alpha, \beta]) = \prod g_{[v_i,v_{i+1}]} \mapsto 0.$$

Further reading: abelianization,

Example 3.5. Let K be a tetrahedron, $|K| \cong S^2$, then

$$H_0(K) = \mathbb{Z}, \quad H_1(K) = \{0\}, \quad H_2(K) = .$$

Note that

$$0 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \to 0,$$

where $C_2 = \mathbb{Z}^4$, $C_1 = \mathbb{Z}^6$. Then

$$H_2(K) = Z_2 = \ker \partial.$$

$$\begin{aligned} [v_0, v_1, v_2] &\mapsto [v_1, v_2] - [v_0, v_2] + [v_0, v_1], \\ [v_0, v_2, v_3] &\mapsto [v_2, v_3] - [v_0, v_3] + [v_0, v_2], \\ [v_0, v_3, v_1] &\mapsto [v_3, v_1] - [v_0, v_1] + [v_0, v_3], \\ [v_1, v_2, v_3] &\mapsto [v_2, v_3] - [v_1, v_3] + [v_1, v_2]. \end{aligned}$$

In matrix form,

$$\begin{bmatrix} [v_0, v_1, v_2] \\ [v_0, v_2, v_3] \\ [v_0, v_3, v_1] \\ [v_1, v_2, v_3] \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} [v_1, v_2] \\ [v_0, v_2] \\ [v_0, v_1] \\ [v_0, v_3] \\ [v_2, v_3] \\ [v_3, v_1] \end{bmatrix}$$

Let $\Delta^{(n+1)}$: a n+1-simplex consists of $v_0, v_1, \ldots, v_{n+1}$, $S = \{(v_{k_0}, v_{k_1}, \ldots, v_{k_d}), v_{k_i} \in \{v_0, \ldots, v_{n+1}\}\}$. Let $K^{(n+1)}$: a simplex consists of $v_0, v_1, \ldots, v_{n+1}$ without the simplex of all vertices of $v_0, v_1, \ldots, v_{n+1}$, $S = \{(v_{k_0}, v_{k_1}, \ldots, v_{k_d}), v_{k_i} \in \{v_0, \ldots, v_{n+1}\}, d \leq n\}$.

$$H_0(K^{(n)}) = \mathbb{Z},$$

$$0 \xrightarrow{\partial} C_{n+1}(K^{(n)}) \xrightarrow{\partial} C_n(K^{(n)}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{q+1}(K^{(n)}) \xrightarrow{\partial} C_q(K^{(n)}) \xrightarrow{\partial} C_{q-1}(K^{(n)}) \xrightarrow{\partial} \cdots$$
$$0 \xrightarrow{\partial} C_{n+1}(\Delta^{(n+1)}) \xrightarrow{\partial} C_n(\Delta^{(n+1)}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{q+1}(\Delta^{(n+1)}) \xrightarrow{\partial} C_q(\Delta^{(n+1)}) \xrightarrow{\partial} C_{q-1}(\Delta^{(n+1)}) \xrightarrow{\partial} \cdots$$

where $C_q(K^{(n)})$ and $C_q(\Delta^{(n+1)})$ are the free modules spanned by q-dimensional simplexes of $K^{(n)}$ and $\Delta^{(n+1)}$, respectively. Note that $C_q(K^{(n)}) = C_q(\Delta^{(n+1)})$ for $q \leq n$, therefore $H_q(K^{(n)}) = H_q(\Delta^{(n+1)}), q \leq n-1$.
What is $H_q(\Delta^{(n+1)})$? Note that $\Delta^{(n+1)}$ can be obtained from $\Delta^{(n)}$ by applying add-one-dimension trick, in other words, $\Delta^{(n+1)} = C\Delta^{(n)}$.

Definition 3.6. Let L be a simplicial complex, the cone of L, CL consists of simplexes $[v, v_0, v_1, \ldots, v_n]$, where $[v_0, v_1, \ldots, v_n]$ is a simplex of L.

Lemma 3.4. $H_q(CL) = \{0\}, q \geq 1$.

Proof. Consider

$$d : [v_0, v_1, \ldots, v_q] \mapsto \begin{cases} [v, v_0, v_1, \ldots, v_q] & \text{if } [v_0, v_1, \ldots, v_q] \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \partial d[v_0, v_1, \ldots, v_q] &= \partial[v, v_0, v_1, \ldots, v_q] \\ &= [v_0, v_1, \ldots, v_q] - \sum_{i=0}^{q} (-1)^i [v, v_0, \ldots, v_i, \ldots, v_q]. \end{aligned}$$

and

$$\begin{aligned} d\partial[v_0, v_1, \ldots, v_q] &= d \sum_{i=0}^{q} (-1)^i [v_0, \ldots, v_i, \ldots, v_q] \\ &= \sum_{i=0}^{q} (-1)^i d[v_0, \ldots, v_i, \ldots, v_q] \\ &= \sum_{i=0}^{q} (-1)^i [v, v_0, \ldots, v_i, \ldots, v_q]. \end{aligned}$$

Observe that $(\partial d + d\partial)[v_0, \ldots, v_q] = [v_0, \ldots, v_q]$. Therefore, there exists a map such that

$$(\partial d + d\partial)(\sigma) = \sigma, \forall \sigma \in C_q(CL).$$

If $\sigma \in Z_q(CL)$, i.e., $\partial\sigma = 0$. Then

$$(\partial d + d\partial)(\sigma) = \partial(d\sigma) + d(\partial\sigma) = \partial(d\sigma) = \sigma \implies \sigma \in B_q(CL).$$

Hence $H_q(\Delta^{(n+1)}) = H_q(K^{(n)}), q \leq n-1$. And $H_n(K^{(n)}) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = \ker \partial_n = \mathbb{Z}$. In a nutshell,

$$H_i(K^{(n)}) = \begin{cases} \mathbb{Z} & i = 0, n, \\ 0 & i = 1, \ldots, n-1. \end{cases}$$

□

Theorem 3.8.

(a) $H_i(\Sigma) = H_i(\Sigma^m), i = 0, 1, \ldots,$ where $H_i(|\Sigma|)$ is well-defined.

(b) Let $f : |\Sigma| \to |\Pi|$ continuous, $\exists f_{*,i} : H_i(|\Sigma|) \to H_i(|\Pi|)$ such that

$$|\Sigma| \stackrel{\varphi}{\to} |\Pi| \stackrel{\psi}{\to} |\Omega|, |\Sigma| \stackrel{\varphi\psi}{\to} |\Omega|, \quad H_i(|\Sigma|) \stackrel{\varphi_{*,i}}{\to} H_i(|\Pi|) \stackrel{\psi_{*,i}}{\to} H_i(|\Omega|), \quad H_i(|\Sigma|) \stackrel{\varphi_{*,i}\psi_{*,i}}{\to} H_i(|\Omega|),$$

(c) $\varphi \sim \psi$, then $(\varphi)_{*,i} = (\psi)_{*,i}$.

Theorem 3.9. S^m, S^n are not of the same homotopy type if $m \neq n$.

Proof.

$$H_i(S^m) \neq H_i(S^n).$$

□

Theorem 3.10. $\mathbb{R}^m$, $\mathbb{R}^n$ are not homeomorphic.

Theorem 3.11 (Brouwer fixed point theorem)**.** If $f : D^n \to D^n$ continuous, then $\exists x^* \in D^n$ such that

$$f(x^*) = x^*.$$

Proof. Suppose that there is no fixed point, and assume that there is a conitnuous map $\varphi : D^n \to \partial D^n$ such that $\varphi|_{\partial D^n} = \text{id}$. Then use $(n-1)$th homology group in place of the fundamental group for the case $n = 2$. □

4 Degree and Lefschetz Number

Definition 4.1. A map from the n-sphere to itself is called the degree, i.e., $f : S^n \to S^n$. $(f)_n : H_n(S^n) \to H_n(S^n) \iff \mathbb{Z} \to \mathbb{Z} \iff 1 \mapsto (f)_n(1) \in \mathbb{Z}$.

Example 4.1.

- $S^1 \to S^1 = \{z \in \mathbb{Z} : |z| = 1\}$, and $f : z \mapsto z^n$, then $\deg(z \mapsto z^n) = n$.
- $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$, $\theta : (x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, -x_2, \ldots, -x_{n+1})$, then $\deg(\theta) = (-1)^{n+1}$. Let

  $$\theta_k : (x_1, x_2, \ldots, x_k, \ldots, x_{n+1}) \mapsto (x_1, x_2, \ldots, -x_k, \ldots, x_{n+1}),$$

  and $\deg(\theta_k) = -1$. Therefore,

  $$\deg(\theta) = \deg(\theta_{n+1}\theta_n \cdots \theta_1) = \deg(\theta_{n+1}) \deg(\theta_n) \cdots \deg(\theta_1) = (-1)^{n+1}.$$

Remark 4.1.

$$\deg(f \circ g) = \deg(f) \cdot \deg(g), \quad f \sim g \implies \deg(f) = \deg(g).$$

Corollary 4.1. If $f : S^n \to S^n$ has *no* fixed point, then $\deg(f) = (-1)^{n+1}$.

Proof. Suppose that $f(x) \neq x$, then

$$H(x,t) = \frac{tf(x) + (1-t)(-x)}{\|tf(x) + (1-t)(-x)\|_2}.$$

Then $f \overset{H}{\sim} \theta$. □

Corollary 4.2. Let n be even, if $f : S^n \to S^n$ is homotopic to the identity map. Then f must have fixed points.

Proof. It is known that

$$\deg(f) = \operatorname{def}(\mathrm{id}) = 1.$$

If f has no fixed point, then

$$\deg(f) = (-1)^{n+1} = -1.$$

□

4.1 Dynamical System

Let G be a discrete group. Assume that G acting on S^n freely, i.e., $\forall g \in G \backslash \{e\}$, $g(x) \neq x, \forall x \in S^n$. $g, h \in G\backslash\{e\}$, g, h have no fixed points, therefore, $\deg(g) = -1, \deg(h) = -1$. Therefore, $\deg(g \circ h) = \deg(g) \cdot \deg(h) = 1$. However, if $g \circ h \neq \mathrm{id}$, then $\deg(g \circ h) = -1$. Contradiction. Therefore, $g \circ h = \mathrm{id} \implies h = g^{-1}$. Also, $g \circ g = \mathrm{id}$, therefore, $G = \mathbb{Z}_2$.
Let $f : S^n \to S^n$. How do we calculate $\deg(f)$? Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is a simplicial complex consists of n-simplexes. Then we try to find a simplicial map $S : |\Sigma^{(m)} \to |\Sigma|$ to approximate $f : |\Sigma| \to |\Sigma|$. First, find the number α of simplexes of $\Sigma^{(m)}$ that are sent to σ, and the number β of simplexes of $\Sigma^{(m)}$ that are sent to $-\sigma$, then $\deg(f) = \alpha - \beta$.
Finitely generated abelian groups are cartesian product of $\mathbb{Z}, \mathbb{Z}_2, \ldots$: $(\mathbb{Z})^m \oplus$
Let K be a simplicial complex, and $|K^{(0)}|, |K^{(1)}|, |K^{(2)}|, \ldots, |K^{(n)}|, \ldots$ be the number of 0-simplex (vertex), 1-simplex (edge), 2-simplex (face), ..., n-simplex,... denoted by $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}, \ldots$. And the homology of K is as follows,

$$\begin{matrix} H^{(0)}(K) \\ H^{(1)}(K) \\ H^{(2)}(K) \\ \vdots \\ H^{(n)}(K) \end{matrix} = \begin{matrix} \mathbb{Z}^{\beta_0} \oplus \mathrm{Tor} \\ \mathbb{Z}^{\beta_1} \oplus \mathrm{Tor} \\ \mathbb{Z}^{\beta_2} \oplus \mathrm{Tor} \\ \vdots \\ \mathbb{Z}^{\beta_n} \oplus \mathrm{Tor} \end{matrix},$$

where $\beta_i = \begin{cases} 1 & i \text{ is even,} \\ 0 & i \text{ is odd.} \end{cases}$, $\mathbb{Z}^{\beta_i}$ is torsion-free part, $i = 0, 1, \ldots$.

Theorem 4.1.

$$\chi(K) = \sum_{n=0}^{\infty} (-1)^n \alpha^{(n)} = \sum_{n=0}^{\infty} (-1)^n \beta^{(n)}.$$

Corollary 4.3. If $|K_1| \sim |K_2|$, then $\chi(K_1) = \chi(K_2)$.

Note that

$$G = \mathbb{Z}^n \oplus \text{Tor}.$$

Then

$$G \otimes \mathbb{Q} \cong (\mathbb{Z}^n \oplus \text{Tor}) \otimes \mathbb{Q} = (\mathbb{Z}^n \times \mathbb{Q}) \oplus (\text{Tor} \otimes \mathbb{Q}) = \mathbb{Q}^n.$$

Therefore, $\dim(G \otimes \mathbb{Q}) = n$. Then $\beta_n = \dim_{\mathbb{Q}}(H^{(n)}(K) \otimes \mathbb{Q})$. Let $C_n \otimes \mathbb{Q}$ be the vector space generated by n-simplexes. Then $H_n(\cdot, \mathbb{Q}) = \frac{\ker(\partial)}{\text{Im}(\partial)}$.

Lemma 4.1. $\beta_n = \dim(H_n(\cdot, \mathbb{Q}))$.

Then

$$\beta_q = \dim_{\mathbb{Q}}(H_q(\cdot, \mathbb{Q})) = \dim_{\mathbb{Q}}\left(\frac{\ker(\partial)}{\text{Im}(\partial)}\right) = \dim_{\mathbb{Q}}\left(\frac{Z_q}{B_q}\right) = \dim_{\mathbb{Q}}(Z_q) - \dim_{\mathbb{Q}}(B_q),$$

where Z_q is rational cycle and B_q is the boundary.
Let C_q be the chain, then $C_{q+1} = \langle C_1^{(q+1)}, C_2^{(q+1)}, C_{\alpha_{q+1}}^{(q+1)} \rangle$. Now let's calculate α_q,

$$\begin{aligned}
\alpha_q &= \gamma_q + \dim(Z_q) \\
&= \gamma_q + (\dim(Z_q) - \dim(B_q)) + \dim(B_q) \\
&= \gamma_q + \beta_q + \gamma_{q+1}.
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_0 &= \gamma_0 + \beta_0 + \gamma_1 \\
\alpha_1 &= \gamma_1 + \beta_1 + \gamma_2 \\
\alpha_2 &= \gamma_2 + \beta_2 + \gamma_3 \\
&\vdots \\
\alpha_n &= \gamma_n + \beta_n + \gamma_{n+1}
\end{aligned}$$

constructing the telescoping series gives

$$\sum_{i=0}^{n} (-1)^i \alpha_i = \sum_{i=0}^{n} (-1)^i \beta_i,$$

because γ_0 and γ_{n+1} are zero.

4.2 Lefschetz Number

Let X be triangulable space, $f : X \to X$. Consider $H_q(X, \mathbb{Q})$, which is a finite-dimensional vector space over $\mathbb{Q}$. Let $f_{*\cdot q} : H_q(X, \mathbb{Q}) \to H_q(X, \mathbb{Q})$, where $H_q(X, \mathbb{Q}) \cong \mathbb{Q}^n$.

Definition 4.2. $\Lambda_f = \sum_q (-1)^q \operatorname{tr}(f_{*\cdot q})$.

Remark 4.2. If $f \sim g$, then $\Lambda_f = \Lambda_g$.

Example 4.2. Consider $\text{id} : X \to X$. What is Λ_{id}. $\text{id}_{*\cdot q} : H_q(X, \mathbb{Q}) \to H_q(X, \mathbb{Q})$, we have $\text{id}_{*\cdot q} = \text{id}$. Therefore,

$$\operatorname{tr}(\text{id}) = \dim_{\mathbb{Q}}(H_q(X, \mathbb{Q})) = \beta_q.$$

Then

$$\Lambda_{\text{id}} = \sum_{q=0}^{\infty} (-1)^q \beta_q = \sum_{q=0}^{\infty} (-1)^q \alpha_q = \chi(X),$$

where α_q is the number of q-simplexes.

Example 4.3. Consider $pt : X \to \{x\} \curvearrowright X$. For $q \geq 1$: $(pt)_{*\cdot q} : H_q(X, \mathbb{Q}) \to H_q(\{x\}, \mathbb{Q}) = 0 \to H_q(X, \mathbb{Q})$, then $(pt)_{*\cdot q}$ is zero map. Therefore $\operatorname{tr}(pt)_{*,q} = 0$ for $q \geq 1$. However, when $q = 0$, $\operatorname{tr}(pt)_{*\cdot q} = 1$. As a result, $\Lambda_{\text{pt}} = 1$.

Example 4.4. Consider $f : S^n \to S^n$. We have

$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

And we can obtain

$$H_q(S^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $(f)_{*\cdot n} : H_n(S^n) = \mathbb{Z} \to H_n(S^n) = \mathbb{Z}$. Hence, $(f_*)_n : H_n(S^n, \mathbb{Q}) = \mathbb{Q} \to H_n(S^n, \mathbb{Q}) = \mathbb{Q}$, that is, $1 \cdot r \to \deg(f) \cdot r$. Hence, $\Lambda_f = 1 + (-1)^n \deg(f)$.

Theorem 4.2 (Lefschetz fixed-point theorem)**.** Let X be a triangulable space and $f : X \to X$. Then

$$\Lambda_f \neq 0 \implies f \text{ must have a fixed point.}$$

Corollary 4.4. Let X be a triangulable space and $f : X \to X$. Assume that f is homotopic to identity map. Then

$$\chi(X) \neq 0 \implies f \text{ must have a fixed point.}$$

Corollary 4.5. Let X be a triangulable space and $f : X \to X$. Assume that f is homotopically trivial, that is, f is contractable. Then

$$\Lambda_f = 1 \implies f \text{ must have a fixed point.}$$

Corollary 4.6. If $\deg(f) \neq \pm 1$, then f must have a fixed point.

Assume that f is a simplicial map. Assume that f has no fixed point, we can show that

$$\sum_q (-1)^q \operatorname{tr}(f_q) = 0.$$

For k large, for any simplex σ in $X^{(k)}$,

$$\operatorname{dist}(f(\sigma), \sigma) > 0.$$