Full Name: ______________________________ **W-Number:** ____________________

MATH 2205: Calculus II – Sample Exam 3 Solution

Summer 2019 – Friday, July 05, 2019

Instructions:

- Show all your work and use the space provided on the exam. Correct mathematical notation is required and all partial credit is at discretion of the grader.
- Write neatly and make sure your work is organized.
- Make certain that you have written your Full Name and W-Number in the spaces provided at the top of the exam. Failure to do so may result in a loss of points.
- No aids beyond a scientific, non-graphing calculator are allowed. This means no notes, no cell phones, etc., are permitted during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 10 pages. Please check to see that your copy has all the pages.

For Instructor Use Only

Question	1	2	3	4	5	6	7	8	9	Total
Points	15	20	10	10	10	10	10	15	10	100
Mark										

Formulae you may find useful:

- $\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$
- $\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.
- $\int x^{-1} \, dx = \ln|x| + C.$
- $\int e^x \, dx = e^x + C.$
- $\int \sin x \, dx = -\cos x + C.$
- $\int \cos x \, dx = \sin x + C.$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C.$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C.$
- $\int \sec x \tan x \, dx = \sec x + C.$
- $\int \sec^2 x \, dx = \tan x + C.$
- $\cos^2 \theta = \frac{1+\cos 2\theta}{2}.$
- $\sin^2 \theta = \frac{1-\cos 2\theta}{2}.$
- $\sin^2 \theta + \cos^2 \theta = 1.$
- $\sin 2\theta = 2 \sin \theta \cos \theta.$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$
- Midpoint Rule: $M(n) = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) \Delta x.$
- Trapezoid Rule: $T(n) = \left[\frac{1}{2} f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2} f(x_n)\right] \Delta x.$
- Simpson's Rule: $S(n) = \sum_{k=0}^{n/2-1} [f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})] \frac{\Delta x}{3}.$

1. (15 points) Circle TRUE if the statement is true or FALSE if it is not, and justify your choice briefly.

   (a) $\boxed{\text{TRUE}}$ or FALSE: Integration by Parts can be used to write $\displaystyle\int x^2e^{-x}\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx$.

   Explanation: Let $u = x^2 \implies du = 2x\,dx; dv = e^{-x}\,dx \implies v = -e^{-x}$. Then we apply the Integration by Parts, i.e., $\displaystyle\int u\,dv = uv - \int v\,du$,

   $$\int x^2e^{-x}\,dx = x^2(-e^{-x}) - \int(-e^{-x})(2x)\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx.$$

   (b) $\boxed{\text{TRUE}}$ or FALSE: If the interval of convergence of the series $\displaystyle\sum_{k=1}^{\infty} c_kx^k$ is $(-3,3)$, then the interval of convergence of $\displaystyle\int\left(\sum_{k=1}^{\infty} c_kx^k\right)dx$ is also $(-3,3)$.

   Explanation:

   $$\int\sum_{k=1}^{\infty} c_kx^k\,dx = \sum_{k=1}^{\infty} c_k\int x^k\,dx = \sum_{k=1}^{\infty} c_k\frac{x^{k+1}}{k+1} + C.$$

   By integrating the series term by term, the interval of convergence remains unchanged (this can also be shown by Ratio Test). The endpoints of the interval depend on c_k.

   (c) TRUE or $\boxed{\text{FALSE}}$: The Fundamental Theorem of Calculus uses the derivative of the function f to evaluate the definite integral $\displaystyle\int_a^b f(x)\,dx$.

   Explanation: FTOC uses antiderivative.

   (d) $\boxed{\text{TRUE}}$ or FALSE: If $\displaystyle\sum_{k=0}^{\infty}|a_k|$ converges, then $\displaystyle\sum_{k=0}^{\infty}(\sin k\cdot a_k)$ must converge.

   Explanation: Note that $\sin k\cdot a_k \le |\sin k\cdot a_k| = |\sin k||a_k| \le |a_k|$ because $|\sin k| \le 1$. Then by Comparison Test, if $\displaystyle\sum_{k=0}^{\infty}|a_k|$ converges, then $\displaystyle\sum_{k=0}^{\infty}\sin k\cdot a_k$ must converge.

   (e) TRUE or $\boxed{\text{FALSE}}$: $\dfrac{[3(k+1)]!}{(3k)!} = 3k+1$.

   Explanation:

   $$\frac{[3(k+1)]!}{(3k)!} = \frac{(3k+3)!}{(3k)!} = \frac{(3k)!\,(3k+1)(3k+2)(3k+3)}{(3k)!} = (3k+1)(3k+2)(3k+3).$$

2. (20 points) Evaluate the following sums.

(a) $\displaystyle\sum_{k=10}^{100}(k+5)(k+4)$.

Solution.

$$
\begin{aligned}
&\sum_{k=10}^{100}(k+5)(k+4)\\
=\;&\sum_{k=10}^{100}(k^2+9k+20)\\
=\;&\sum_{k=1}^{100}(k^2+9k+20)-\sum_{k=1}^{9}(k^2+9k+20)\\
=\;&\left[\sum_{k=1}^{100}k^2+9\sum_{k=1}^{100}k+\sum_{k=1}^{100}20\right]-\left[\sum_{k=1}^{9}k^2+9\sum_{k=1}^{9}k+\sum_{k=1}^{9}20\right]\\
=\;&\frac{n(n+1)(2n+1)}{6}+9\cdot\frac{n(n+1)}{2}+20n\Big|_{n=9}^{n=100}\\
=\;&\left(\frac{100\cdot 101\cdot 201}{6}+9\cdot\frac{100\cdot 101}{2}+20\cdot 100\right)-\left(\frac{9\cdot 10\cdot 19}{6}+9\cdot\frac{9\cdot 10}{2}+20\cdot 9\right)\\
=\;&384930.
\end{aligned}
$$

$\square$

(b) $\displaystyle\sum_{k=0}^{\infty}\frac{5}{(5k+1)(5k+6)}$.

Solution.

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{5}{(5k+1)(5k+6)}\\
=\;&\sum_{k=0}^{\infty}\frac{(5k+6)-(5k+1)}{(5k+1)(5k+6)}\\
=\;&\sum_{k=0}^{\infty}\left(\frac{1}{5k+1}-\frac{1}{5k+6}\right)\\
=\;&\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{1}{5k+1}-\frac{1}{5k+6}\right)\\
=\;&\lim_{n\to\infty}1-\frac{1}{6}+\frac{1}{6}-\frac{1}{11}+\frac{1}{11}-\frac{1}{16}+\cdots+\frac{1}{5n-4}-\frac{1}{5n+1}+\frac{1}{5n+1}-\frac{1}{5n+6}\\
=\;&\lim_{n\to\infty}1-\frac{1}{5n+6}\\
=\;&1.
\end{aligned}
$$

$\square$

(c) $\frac{3}{7} + \frac{6}{21} + \frac{12}{63} + \frac{24}{189} + \cdots$.

SOLUTION. Observe that the series is a geometric series with the ratio $r = \frac{a_{k+1}}{a_k} = \frac{6/21}{3/7} = \frac{12/63}{3/7} = \cdots = \frac{2}{3}$, $a = \frac{3}{7}$, $m = 0$. Then

$$\begin{aligned}
\frac{3}{7} + \frac{6}{21} + \frac{12}{63} + \frac{24}{189} + \cdots &= \sum_{k=0}^{\infty} \frac{3}{7}\left(\frac{2}{3}\right)^k && [a = \frac{3}{7}, r = \frac{2}{3}, m = 0] \\
&= \frac{\frac{3}{7}\left(\frac{2}{3}\right)^0}{1 - \frac{2}{3}} && [\sum_{k=m} ar^k = \frac{ar^m}{1-r}] \\
&= \frac{3}{7}\frac{3}{1} \\
&= \frac{9}{7}.
\end{aligned}$$

□

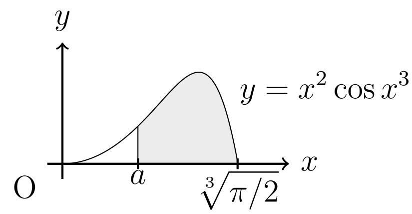
(d) $\sum_{k=0}^{\infty} \left(e^{-2k+1} + \frac{1}{\pi^{3k-1}}\right)$.

SOLUTION.

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(e^{-2k+1} + \frac{1}{\pi^{3k-1}}\right) &= \sum_{k=0}^{\infty} e^{-2k+1} + \sum_{k=0}^{\infty} \frac{1}{\pi^{3k-1}} \\
&= \sum_{k=0}^{\infty} e \cdot (e^{-2})^k + \sum_{k=0}^{\infty} \pi \left(\frac{1}{\pi^3}\right)^k \\
&= \frac{e \cdot (e^{-2})^0}{1 - e^{-2}} + \frac{\pi(\pi^{-3})^0}{1 - \pi^{-3}} && [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}] \\
&= \frac{e}{1 - e^{-2}} + \frac{\pi}{1 - \pi^{-3}} \\
&= \frac{e^3}{e^2 - 1} + \frac{\pi^4}{\pi^3 - 1}.
\end{aligned}$$

□

3. (10 points) Determine the value of the positive parameter a so that the area of the shaded region in the picture is equal to $\frac{1}{3}$.



SOLUTION. Let $f(x) = x^2 \cos x^3$. The area of the shaded region is

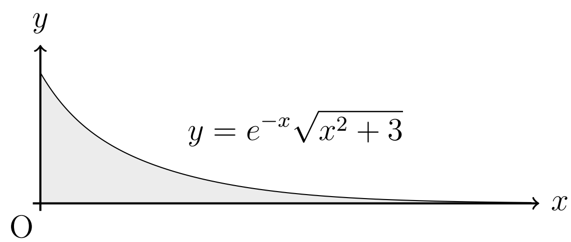
$$\begin{aligned}
A &= \int_a^b f(x)\,dx \\
&= \int_a^{\sqrt[3]{\pi/2}} x^2 \cos x^3\,dx \\
&= \frac{1}{3}\int_a^{\sqrt[3]{\pi/2}} \cos x^3 (3x^2)\,dx \\
&= \frac{1}{3}\int_{u(a)=a^3}^{u(\sqrt[3]{\pi/2})=\pi/2} \cos u\,du \qquad [u = x^3, du = 3x^2\,dx] \\
&= \frac{1}{3}\sin u\Big|_{a^3}^{\pi/2} \\
&= \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin a^3\right) \\
&= \frac{1}{3} - \frac{1}{3}\sin a^3.
\end{aligned}$$

Given that the area of the shaded region $A = \frac{1}{3}$, we have

$$\frac{1}{3} - \frac{1}{3}\sin a^3 = \frac{1}{3} \implies \sin a^3 = 0 \implies a^3 = 0 \implies a = 0.$$

$\square$

4. (10 points) Consider the infinitely long shaded region R indicated in the picture. Determine the volume of the solid of the revolution obtained when R is revolved about the x-axis.



SOLUTION. We can use the Disk Method to calculate the volume of solid of the revolution. Let $f(x) = e^{-x}\sqrt{x^2+3} = e^{-x}(x^2+3)^{1/2}$, then

$$
\begin{aligned}
V &= \int_a^b \pi f(x)^2\,dx \\
&= \int_0^\infty \pi[e^{-x}(x^2+3)^{1/2}]^2\,dx \\
&= \pi \lim_{b\to\infty}\int_0^b e^{-2x}(x^2+3)\,dx \\
&= \pi\left(\lim_{b\to\infty}\int_0^b e^{-2x}x^2\,dx + \lim_{b\to\infty}\int_0^b 3e^{-2x}\,dx\right) \\
&= \pi\left(\lim_{b\to\infty}\frac{e^{-2x}}{-2}x^2\Big|_0^b - \int_0^b \frac{e^{-2x}}{-2}(2x)\,dx + \lim_{b\to\infty}3\frac{e^{-2x}}{-2}\Big|_0^b\right) \\
&= \pi\left[\lim_{b\to\infty}\frac{e^{-2b}}{-2}b^2 - \frac{e^0}{-2}\cdot 0^2 + \int_0^b e^{-2x}x\,dx + \lim_{b\to\infty}3\left(\frac{e^{-2b}}{-2} - \frac{e^0}{-2}\right)\right] \\
&= \pi\left(\lim_{b\to\infty}\frac{e^{-2x}}{-2}x\Big|_0^b - \int_0^b\frac{e^{-2x}}{-2}\,dx + \frac{3}{2}\right) \\
&= \pi\left(\lim_{b\to\infty}\frac{e^{-2b}}{-2}b - \frac{e^0}{-2}\cdot 0 + \frac{1}{2}\int_0^b e^{-2x}\,dx + \frac{3}{2}\right) \\
&= \pi\left(\lim_{b\to\infty}\frac{1}{2}\frac{e^{-2x}}{-2}\Big|_0^b + \frac{3}{2}\right) \\
&= \pi\left[\lim_{b\to\infty}\frac{1}{2}\left(\frac{e^{-2b}}{-2} - \frac{e^0}{-2}\right) + \frac{3}{2}\right] \\
&= \pi\left[\frac{1}{2}\left(0 + \frac{1}{2}\right) + \frac{3}{2}\right] \\
&= \frac{7\pi}{4}.
\end{aligned}
$$

□

5. (10 points) Find the interval of convergence and radius of convergence for the power series given by

$$\sum_{k=1}^{\infty} \frac{(-1)^k (x-3)^k}{k^{2/3}}.$$

Solution. Since the Ratio/Root Test requires positive terms. Here we test the absolute convergence of the given series using Ratio Test. We can identify that $a_k = \dfrac{(-1)^k (x-3)^k}{k^{2/3}}$. Then

$$\begin{aligned}
r &= \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \\
&= \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(x-3)^{k+1}/(k+1)^{2/3}}{(-1)^k (x-3)^k / k^{2/3}} \right| \\
&= \lim_{k\to\infty} \left| \frac{(-1)^k \cdot (-1)(x-3)^k \cdot (x-3)}{(k+1)^{2/3}} \frac{k^{2/3}}{(-1)^k (x-3)^k} \right| \\
&= \lim_{k\to\infty} \left| \frac{(-1)(x-3)k^{2/3}}{(k+1)^{2/3}} \right| \\
&= \lim_{k\to\infty} |x-3| \left(\frac{k}{k+1} \right)^{2/3} \\
&= |x-3| \lim_{k\to\infty} \left(\frac{1}{1+1/k} \right)^{2/3} \\
&= |x-3|.
\end{aligned}$$

The Ratio Test requires $r = |x-3| < 1$, then we have

$$|x-3| < 1 \implies -1 < x-3 < 1 \implies 2 < x < 4.$$

However, the Ratio (Root) Test is inconlusive when r (ρ) is 1. Therefore, we need to check the endpoints of the interval of convergence $x = 2$ and $x = 4$. When $x = 2$, the series becomes

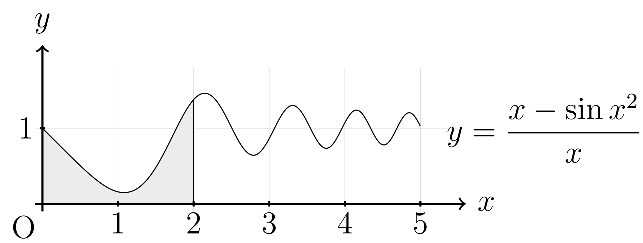
$$\sum_{k=1}^{\infty} \frac{(-1)^k (2-3)^k}{k^{2/3}} = \sum_{k=1}^{\infty} \frac{(-1)^k (-1)^k}{k^{2/3}} = \sum_{k=1}^{\infty} \frac{[(-1)^2]^k}{k^{2/3}} = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}},$$

which is a p-series with $p = \dfrac{2}{3} < 1$. Hence when $x = 2$, the series diverges. When $x = 4$, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^k (4-3)^k}{k^{2/3}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2/3}},$$

which is an alternating series. We can perform the Alternating Series Test, noting $\dfrac{1}{k^{2/3}}$ is decreasing as k increases, and $\lim_{k\to\infty} \frac{1}{k^{2/3}} = 0$. Therefore, the series converges when $x = 4$. Thus, the interval of convergence is $(2, 4]$, i.e., $2 < x \le 4$. Therefore, the radius of convergence is $R = (4-2)/2 = 1$. $\square$

6. (10 points) Use MacLaurin Series to approximate the net area between the function $y = \dfrac{x - \sin x^2}{x}$ and the x-axis from $x = 0$ to $x = 2$ with an error no greater than $10^{-4} = 0.0001$. Be sure to justify that your error satisfies the given bound.



SOLUTION. Let $f(x) = \frac{x - \sin x^2}{x}$, then the area of the region is

$$A = \int_a^b f(x)\,dx = \int_0^2 \frac{x - \sin x^2}{x}\,dx = \int_0^2 dx - \int_0^2 x^{-1}\sin x^2\,dx = 2 - \int_0^2 x^{-1}\sin x^2\,dx.$$

Recall that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \implies \sin x^2 = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}.$$

Then the integral becomes

$$\begin{aligned} A = 2 - \int_0^2 x^{-1}\sin x^2\,dx &= 2 - \int_0^2 x^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}\,dx. \\ &= 2 - \int_0^2 \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2} \cdot x^{-1}}{(2k+1)!}\,dx. \\ &= 2 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^2 x^{4k+1}\,dx. \\ &= 2 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{4k+2}}{4k+2}\Bigg|_0^2 \\ &= 2 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{2^{4k+2}}{4k+2}. \end{aligned}$$

Noting that the second term is an alternating series, then we need to find n such that the partial sum $S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \frac{2^{4k+2}}{4k+2}$ has error less than 10^{-4}, then

$$|R_n| < a_{n+1} = \frac{2^{4(n+1)+2}}{[2(n+1)+1]! \cdot [4(n+1)+2]} < 10^{-4}.$$

Solve for n, we have $n > 5$, then we pick $n = 6$, which gives $S_6 = \sum_{k=0}^{6} \frac{(-1)^k}{(2k+1)!} \frac{2^{4k+2}}{4k+2} \approx 0.8791276$. Then $A \approx 2 - S_6 \approx 1.1208724$. $\square$

7. (10 points) Find power series representations centered at 0 for $\ln\left(\frac{1+8x^3}{1-x^3}\right)$ and give its interval of convergence.

Solution. First, we can rewrite $\ln\left(\frac{1+8x^3}{1-x^3}\right)$ as follows,

$$\ln\left(\frac{1+8x^3}{1-x^3}\right) = \ln(1+8x^3) - \ln(1-x^3) = \int \frac{24x^2}{1+8x^3}\,dx - \int \frac{-3x^2}{1-x^3}\,dx.$$

Recall that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, then

$$\frac{24x^2}{1+8x^3} = 24x^2\frac{1}{1-(-2x)^3} = 24x^2\sum_{k=0}^{\infty}[(-2x)^3]^k = 24\sum_{k=0}^{\infty}(-1)^{3k}2^{3k}x^{3k}\cdot x^2 = 24\sum_{k=0}^{\infty}(-1)^k 2^{3k}x^{3k+2},$$

$$\frac{-3x^2}{1-x^3} = -3x^2\frac{1}{1-x^3} = -3x^2\sum_{k=0}^{\infty}(x^3)^k = -3\sum_{k=0}^{\infty}x^{3k}\cdot x^2 = -3\sum_{k=0}^{\infty}x^{3k+2},$$

which converges on $(-1/2, 1/2)$ and $(-1,1)$, respectively ($|(-2x)^3| < 1 \implies |x| < \frac{1}{2}$ and $|x^3| < 1 \implies |x| < 1$). Then we can substitute the MacLaurin series into the integrals

$$\begin{aligned}
\ln\left(\frac{1+8x^3}{1-x^3}\right) &= \int \frac{24x^2}{1+8x^3}\,dx - \int \frac{-3x^2}{1-x^3}\,dx \\
&= \int 24\sum_{k=0}^{\infty}(-1)^k 2^{3k}x^{3k+2}\,dx - \int (-3)\sum_{k=0}^{\infty}x^{3k+2}\,dx \\
&= 24\sum_{k=0}^{\infty}(-1)^k 2^{3k}\int x^{3k+2}\,dx + 3\sum_{k=0}^{\infty}\int x^{3k+2}\,dx \\
&= 2^3\cdot 3\sum_{k=0}^{\infty}(-1)^k 2^{3k}\frac{x^{3k+3}}{3k+3} + 3\sum_{k=0}^{\infty}\frac{x^{3k+3}}{3k+3} + C \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{(2x)^{3k+3}}{k+1} + \sum_{k=0}^{\infty}\frac{x^{3k+3}}{k+1} + C
\end{aligned}$$

Note that when $x = 0$, $\ln\left(\frac{1+8x^3}{1-x^3}\right)|_{x=0} = \ln 1 = 0$. Then, we have $C = 0$. Hence,

$$\ln\left(\frac{1+8x^3}{1-x^3}\right) = \sum_{k=0}^{\infty}(-1)^k\frac{(2x)^{3k+3}}{k+1} + \sum_{k=0}^{\infty}\frac{x^{3k+3}}{k+1}.$$

Then the interval of convergence is the intersection of $|x| < \frac{1}{2}$ and $|x| < 1$, which is $|x| < \frac{1}{2} \implies -\frac{1}{2} < x < \frac{1}{2}$. We need to check the endpoints $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. When $x = -\frac{1}{2}$, the first term becomes $-\sum_{k=0}^{\infty}\frac{1}{k+1}$ which is a harmonic series and when $x = \frac{1}{2}$, the first term becomes $\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$ which is an alternating harmonic series. It is shown that it diverges at $x = -\frac{1}{2}$ and converges at $x = \frac{1}{2}$. Therefore, the interval of convergence is $(-\frac{1}{2}, \frac{1}{2}]$, i.e., $-\frac{1}{2} < x \le \frac{1}{2}$. $\square$

8. (15 points) Determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{4k^3+1000k-3}{9k^3+20k^2+6}\right)^k.$

Solution. Root Test. Identify that $a_k=\left(\dfrac{4k^3+1000k-3}{9k^3+20k^2+6}\right)^k$. Then

$$\begin{aligned}
\rho=\lim_{k\to\infty}\sqrt[k]{a_k}=\lim_{k\to\infty}(a_k)^{1/k}&=\lim_{k\to\infty}\left[\left(\frac{4k^3+1000k-3}{9k^3+20k^2+6}\right)^k\right]^{1/k}\\
&=\lim_{k\to\infty}\frac{4k^3+1000k-3}{9k^3+20k^2+6}\\
&=\lim_{k\to\infty}\frac{4+1000k^{-2}-3k^{-3}}{9+20k^{-1}+6k^{-3}}\\
&=\frac{4}{9}.
\end{aligned}$$

By Root Test, $\rho=\dfrac{4}{9}<1$, then the given series converges. □

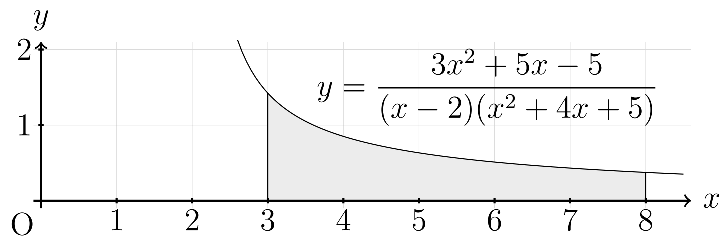
(b) $\displaystyle\sum_{k=1}^{\infty}\frac{(2k)^{2k}}{(2k)!}.$

Solution. Ratio Test. Note that $a_k=\dfrac{(2k)^{2k}}{(2k)!}, a_{k+1}=\dfrac{[2(k+1)]^{2(k+1)}}{[2(k+1)]!}=\dfrac{(2k+2)^{2k+2}}{(2k+2)!}$,

$$\begin{aligned}
r=\lim_{k\to\infty}\frac{a_{k+1}}{a_k}&=\lim_{k\to\infty}\frac{(2k+2)^{2k+2}/(2k+2)!}{(2k)^{2k}/(2k)!}\\
&=\lim_{k\to\infty}\frac{(2k+2)^{2k}(2k+2)^2}{(2k)!\,(2k+1)(2k+2)}\frac{(2k)!}{(2k)^{2k}}\\
&=\lim_{k\to\infty}\frac{(2k+2)^{2k}}{(2k)^{2k}}\frac{(2k+2)^2}{(2k+1)(2k+2)}\\
&=\lim_{k\to\infty}\left(\frac{2k+2}{2k}\right)^{2k}\frac{4k^2+8k+4}{4k^2+6k+2}\\
&=\left[\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\right]^2\cdot\lim_{k\to\infty}\frac{4+8k^{-1}+4k^{-2}}{4+6k^{-1}+2k^{-2}}\\
&=e^2.
\end{aligned}$$

By Ratio Test, $r=e^2>1$, the given series diverges. □

9. (Bonus, 10 points) Compute the area of region R bounded by $y = \dfrac{3x^2+5x-5}{(x-2)(x^2+4x+5)}$, x-axis, $x=3$, $x=8$.



SOLUTION. The area of the region R is

$$A = \int_a^b f(x)\,dx = \int_3^8 \frac{3x^2+5x-5}{(x-2)(x^2+4x+5)}\,dx.$$

Then we apply the partial fraction decomposition to the integrand,

$$\frac{3x^2+5x-5}{(x-2)(x^2+4x+5)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4x+5}.$$

Multiplying both sides by $(x-2)(x^2+4x+5)$ gives

$$3x^2+5x-5 = A(x^2+4x+5)+(Bx+C)(x-2) = (A+B)x^2+(4A-2B+C)x+(5A-2C).$$

Equating like powers of x, we have the following linear equations,

$$\begin{cases} A+B=3 \\ 4A-2B+C=5 \\ 5A-2C=-5 \end{cases} \implies \begin{cases} A=1 \\ B=2 \\ C=5 \end{cases}$$

Then definite integral becomes

$$\begin{aligned}
A = \int_3^8 \frac{3x^2+5x-5}{(x-2)(x^2+4x+5)}\,dx &= \int_3^8 \frac{1}{x-2} + \frac{2x+5}{x^2+4x+5}\,dx \\
&= \int_{u(3)=3-2=1}^{u(8)=8-2=6} \frac{1}{u}\,du + \int_3^8 \frac{2x+4}{x^2+4x+5}\,dx + \int_3^8 \frac{1}{x^2+4x+5}\,dx \\
&= \ln|u|\Big|_1^6 + \int_{v(3)=26}^{v(8)=101} \frac{1}{v}\,dv + \int_3^8 \frac{1}{(x^2+4x+4)+1}\,dx \\
&= \ln 6 + \ln|v|\Big|_{26}^{101} + \int_3^8 \frac{1}{1+(x+2)^2}\,dx \\
&= \ln 6 + \ln 101 - \ln 26 + \arctan(x+2)\Big|_3^8 \\
&= \ln\left(\frac{6\cdot 101}{26}\right) + \arctan 10 - \arctan 5.
\end{aligned}$$

□