Full Name: ______________________ **W-Number:** ______________

MATH 2205: Calculus II – Sample Exam 1 Solution

Summer 2019 – Friday, June 07, 2019

Instructions:

- Show all your work and use the space provided on the exam. Correct mathematical notation is required and all partial credit is at discretion of the grader.
- Write neatly and make sure your work is organized.
- Make certain that you have written your Full Name and W-Number in the spaces provided at the top of the exam. Failure to do so may result in a loss of points.
- No aids beyond a scientific, non-graphing calculator are allowed. This means no notes, no cell phones, etc., are permitted during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 11 pages. Please check to see that your copy has all the pages.

For Instructor Use Only

Question	1	2	3	4	5	6	7	8	Total
Points	15	20	10	10	10	10	10	15	100
Mark									

Formulae you may find useful:

- $\displaystyle\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$
- $\displaystyle\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.
- $\displaystyle\int x^{-1}\,dx = \ln|x| + C.$
- $\displaystyle\int e^x\,dx = e^x + C.$
- $\displaystyle\int \sin x\,dx = -\cos x + C.$
- $\displaystyle\int \cos x\,dx = \sin x + C.$
- $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C.$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C.$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C.$
- $\displaystyle\cos^2\theta = \frac{1+\cos 2\theta}{2}.$
- $\displaystyle\sin^2\theta = \frac{1-\cos 2\theta}{2}.$
- $\sin^2\theta + \cos^2\theta = 1.$
- $\sin 2\theta = 2\sin\theta\cos\theta.$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta.$

1. (15 points) Circle TRUE if the statement is true or FALSE if it is not, and justify your choice briefly.

   (a) TRUE or $\boxed{\text{FALSE}}$: If the function f is always negative on the interval $[a, b]$, then the area and the net area between the curve and the x-axis from a to b are equal.
   Area and net area would have opposite sign.

   (b) TRUE or $\boxed{\text{FALSE}}$: If the average value of a function on $[a, b]$ is positive, then that function cannot be negative on that interval.
   As long as net area is greater than zero, then the average value of the function must be greater than zero.

   (c) $\boxed{\text{TRUE}}$ or FALSE: We could use substitution rule to evaluate $\displaystyle\int_0^1 e^x \cos e^x\, dx$.
   Let $u = e^x$.

   (d) $\boxed{\text{TRUE}}$ or FALSE: Given a region in the first quadrant, it is possible that the volume of the solid obtained by revolving about the x-axis is equal to the volume of the solid obtained by revolving about the y-axis.
   Let $f(x) = x$.

   (e) TRUE or $\boxed{\text{FALSE}}$: The Fundamental Theorem of Calculus uses the derivatives of the function f to evaluate the definite integral $\displaystyle\int_a^b f(x)\, dx$.
   Should use the *antiderivative*.

2. (20 points) Evaluate the following sums and integrals.

(a) $\displaystyle\sum_{k=1}^{10}(2k+3)k.$

Solution.

$$\begin{aligned}\sum_{k=1}^{10}(2k+3)k &= \sum_{k=1}^{10} 2k^2+3k \\ &= 2\sum_{k=1}^{10}k^2+3\sum_{k=1}^{10}k \\ &= 2\frac{n(n+1)(2n+1)}{6}\bigg|_{n=10}+3\frac{n(n+1)}{2}\bigg|_{n=10} \\ &= 770+165 \\ &= 935.\end{aligned}$$

$\square$

(b) $\displaystyle\sum_{k=-2}^{5}(k+1)(k+2)k.$

Solution.

$$\begin{aligned}\sum_{k=-2}^{5}(k+1)(k+2)k = &\ (-2+1)(-2+2)(-2)+(-1+1)(-1+2)(-1) \\ &+(0+1)(0+2)0+\sum_{k=1}^{5}k^3+3k^2+2k \\ = &\ \sum_{k=1}^{5}k^3+3\sum_{k=1}^{5}k^2+2\sum_{k=1}^{5}k \\ = &\ \frac{n^2(n+1)^2}{4}\bigg|_{n=5}+3\frac{n(n+1)(2n+1)}{6}\bigg|_{n=5}+2\frac{n(n+1)}{2}\bigg|_{n=5} \\ = &\ 225+165+30 \\ = &\ 420.\end{aligned}$$

$\square$

(c) $\displaystyle\int_{-3}^{3} 5x^4 + x^2\sin^3(x)\,dx.$

SOLUTION.

$$\begin{aligned}
\int_{-3}^{3} 5x^4 + x^2\sin^3(x)\,dx &= \int_{-3}^{3} 5x^4\,dx + \int_{-3}^{3} x^2\sin^3(x)\,dx \\
&= 2\int_{0}^{3} 5x^4\,dx && [5x^4 \text{ is even and } x^2\sin^3(x) \text{ is odd}] \\
&= 2x^5\Big|_0^3 \\
&= 486.
\end{aligned}$$

□

(d) $\displaystyle\int_{0}^{\pi} e^x\sin^2(2e^x)\,dx.$

SOLUTION.

$$\begin{aligned}
\int_0^\pi e^x \sin^2(2e^x)\,dx &= \frac{1}{2}\int_0^\pi \sin^2(\underbrace{2e^x}_{u})\underbrace{2e^x\,dx}_{du} \\
&= \frac{1}{2}\int_{u(0)}^{u(\pi)} \sin^2(u)du \\
&= \frac{1}{2}\int_{2}^{2e^\pi} \frac{1-\cos 2u}{2}du \\
&= \frac{1}{2}\left(\frac{u}{2}\Big|_2^{2e^\pi} - \frac{1}{4}\int_2^{2e^\pi}\cos\underbrace{2u}_{v}\,\underbrace{2\,du}_{dv}\right) \\
&= \frac{1}{2}\left(\frac{2(e^\pi-1)}{2} - \frac{1}{4}\int_{v(2)}^{v(2e^\pi)}\cos v\,dv\right) \\
&= \frac{1}{2}\left(e^\pi - 1 - \frac{1}{4}\sin v\Big|_4^{4e^\pi}\right) \\
&= \frac{1}{2}\left(e^\pi - 1 - \frac{\sin 4e^\pi - \sin 4}{4}\right) \\
&\approx 11.0999.
\end{aligned}$$

□

3. (10 points) Use right Riemann Sum to approximate the net area of the region bounded by the graph of $f(x) = 3x(x^2+2)+5$ and x-axis on $[0,5]$ for $n=5$.

Solution. Split the interval $[0,5]$ into n subintervals of length Δx, where

$$\Delta x = \frac{b-a}{n} = \frac{5}{5} = 1.$$

For the kth subinterval, we pick $x_k^* = x_k = x_0 + k\Delta x = k$. Then we can approximate the net area of the region bounded by the graph of $f(x) = 3x(x^2+2)+5$ is

$$\begin{aligned}
A &\approx \sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \sum_{k=1}^{5} f(x_k)\Delta x \\
&= \sum_{k=1}^{5} f(k) \\
&= \sum_{k=1}^{5} 3k(k^2+2)+5 \\
&= \sum_{k=1}^{5} 3k^3+6k+5 \\
&= 3\sum_{k=1}^{5} k^3 + 6\sum_{k=1}^{5} k + 5\sum_{k=1}^{5} 1 \\
&= 3\frac{n^2(n+1)^2}{4}\bigg|_{n=5} + 6\frac{n(n+1)}{2}\bigg|_{n=5} + 5n\bigg|_{n=5} \\
&= 675+90+25 \\
&= 790.
\end{aligned}$$

$\square$

4. (10 points) Let $\mathcal{R}$ be the region in the first quadrant between the curves $y = x$ and $y = x^n$, where $n > 1$. Find the value of n so that the area of $\mathcal{R}$ is equal to $\dfrac{1}{10}$.

SOLUTION. First let us determine the interval by finding intersections of the above two curves:

$$x = x^n \implies x(1 - x^{n-1}) = 0 \implies x = 0, x^{n-1} = 1 \implies x = 0, x = 1.$$

Let $f(x) = x$ and $g(x) = x^n$. The area of the region is

$$A = \int_0^1 [f(x) - g(x)]\, dx = \int_0^1 x - x^n\, dx = \frac{x^2}{2} - \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{2} - \frac{1}{n+1}.$$

It is known that $A = 1/10$, then we solve for n as follows

$$\frac{1}{2} - \frac{1}{n+1} = \frac{1}{10} \implies n + 1 = \frac{5}{2} \implies n = \frac{3}{2}.$$

$\square$

5. (10 points) Let R be the region bounded by the following curves: $y = x^2$, $y = 2 - x$, and $x = 0$ in the first quadrant; revolved about the y-axis. Find the volume of the solid.

SOLUTION. First, determine the interval by considering $x^2 = 2 - x$ as follows

$$x^2 = 2 - x \implies x^2 + x - 2 = (x+2)(x-1) = 0 \implies x = 1, x = -2.$$

Note that the curves are in the first quadrant, so $x = 1$ is desired. Let $f(x) = 2 - x$ and $g(x) = x^2$, using shell method, we have

$$\begin{aligned}
V &= \int_0^1 2\pi x[f(x) - g(x)]\,dx \\
&= 2\pi \int_0^1 x[(2-x) - x^2]\,dx \\
&= 2\pi \int_0^1 2x - x^2 - x^3\,dx \\
&= 2\pi \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right) \Bigg|_0^1 \\
&= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) \\
&= \frac{5\pi}{6}.
\end{aligned}$$

$\square$

6. (10 points) Find the arc length of the curve $y = \dfrac{1}{2}(e^x + e^{-x})$ on $[-\ln 2, \ln 2]$ by integrating with respect to x.

SOLUTION. Let $f(x) = \frac{1}{2}(e^x + e^{-x})$. Taking the derivative of $f(x)$ with respect to x then squaring $f'(x)$ yields

$$f'(x) = \frac{1}{2}(e^x - e^{-x}) \implies f'(x)^2 = \left[\frac{1}{2}(e^x - e^{-x})\right]^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x}).$$

Then substitute $f'(x)^2$ into the formula, we have the length of the curve as below

$$\begin{aligned}
L &= \int_{-\ln 2}^{\ln 2} \sqrt{1 + f'(x)^2}\, dx \\
&= \int_{-\ln 2}^{\ln 2} \left[1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})\right]^{1/2} dx \\
&= \int_{-\ln 2}^{\ln 2} \left[\frac{1}{4}(e^{2x} + 2 + e^{-2x})\right]^{1/2} dx \\
&= \int_{-\ln 2}^{\ln 2} \left\{\left[\frac{1}{2}(e^x + e^{-x})\right]^2\right\}^{1/2} dx \\
&= \int_{-\ln 2}^{\ln 2} \frac{1}{2}(e^x + e^{-x})\, dx \\
&= \frac{1}{2}(e^x - e^{-x})\Bigg|_{-\ln 2}^{\ln 2} \\
&= \frac{1}{2}[(e^{\ln 2} - e^{-\ln 2}) - (e^{-\ln 2} - e^{\ln 2})] \\
&= \frac{1}{2}(2e^{\ln 2} - 2e^{-\ln 2}) \\
&= e^{\ln 2} - e^{-\ln 2} \\
&= 2 - \frac{1}{2} \\
&= \frac{3}{2}.
\end{aligned}$$

□

7. (10 points) *Surface area of an ellipsoid* - If the top half of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is revolved about the x-axis, the result is an *ellipsoid* whose axis along the x-axis has length $2a$, whose axis along the y-axis has length $2b$, and whose axis perpendicular to the xy-plane has length $2b$. We assume that $0<b<a$. Find the surface area of the ellipsoid for $a=4$, $b=2$.

Solution. Provided the ellipse equation, we can solve for y in terms of x as follows

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \implies y=b\left(1-\frac{x^2}{a^2}\right)^{1/2}=f(x).$$

Then taking the derivative of $f(x)$ with respect to x using chain rule gives

$$f'(x)=\frac{b}{2}\left(1-\frac{x^2}{a^2}\right)^{-1/2}\left(-\frac{2x}{a^2}\right)=-\frac{bx}{a^2}\left(1-\frac{x^2}{a^2}\right)^{-1/2}.$$

Then we have $f'(x)^2$,

$$f'(x)^2=\left[-\frac{bx}{a^2}\left(1-\frac{x^2}{a^2}\right)^{-1/2}\right]^2=\frac{b^2x^2}{a^4}\left(1-\frac{x^2}{a^2}\right)^{-1}.$$

Then we calculate the surface area by substituting the above into the formula:

$$\begin{aligned}
A&=\int_{-4}^{4}2\pi f(x)\sqrt{1+f'(x)^2}\,dx\\
&=2\pi\int_{-4}^{4}b\left(1-\frac{x^2}{a^2}\right)^{1/2}\left[1+\frac{b^2x^2}{a^4}\left(1-\frac{x^2}{a^2}\right)^{-1}\right]^{1/2}dx\\
&=2\pi b\int_{-4}^{4}\left\{\left(1-\frac{x^2}{a^2}\right)\left[1+\frac{b^2x^2}{a^4}\left(1-\frac{x^2}{a^2}\right)^{-1}\right]\right\}^{1/2}dx\\
&=2\pi b\int_{-4}^{4}\left[\left(1-\frac{x^2}{a^2}\right)+\frac{b^2x^2}{a^4}\right]^{1/2}dx\\
&=4\pi b\int_{0}^{4}\left(1-\frac{x^2}{a^2}+\frac{b^2x^2}{a^4}\right)^{1/2}dx && \text{[the integrand is even]}\\
&=4\pi b\int_{0}^{4}\left(1-\frac{a^2x^2}{a^4}+\frac{b^2x^2}{a^4}\right)^{1/2}dx\\
&=4\pi b\int_{0}^{4}\left(1-\frac{a^2-b^2}{a^4}x^2\right)^{1/2}dx
\end{aligned}$$

Let $c^2=(a^2-b^2)/a^4$, then let $cx=\sin\theta \implies x=\sin\theta/c \implies dx=\cos\theta/c\,d\theta$ and

$\theta = \arcsin cx$. By $\sin^2\theta + \cos^2\theta = 1$, the above integral becomes

$$\begin{aligned}
4\pi b \int_{\arcsin 0}^{\arcsin 4c} \cos\theta \frac{\cos\theta}{c}\, d\theta &= \frac{4\pi b}{c} \int_0^{\arcsin 4c} \cos^2\theta\, d\theta \\
&= \frac{4\pi b}{c} \int_0^{\arcsin 4c} \frac{\cos 2\theta + 1}{2}\, d\theta \\
&= \frac{2\pi b}{c} \int_0^{\arcsin 4c} \cos 2\theta + 1\, d\theta \\
&= \frac{2\pi b}{c} \left(\int_0^{\arcsin 4c} \cos 2\theta\, d\theta + \int_0^{\arcsin 4c} 1\, d\theta \right) \\
&= \frac{2\pi b}{c} \left(\frac{1}{2} \int_0^{\arcsin 4c} \cos \underbrace{2\theta}_{u} \underbrace{2\, d\theta}_{du} + \arcsin 4c \right) \\
&= \frac{2\pi b}{c} \left(\frac{1}{2} \int_{u(0)}^{u(\arcsin 4c)} \cos u\, du + \arcsin 4c \right) \\
&= \frac{2\pi b}{c} \left(\frac{1}{2} \sin u \Big|_0^{2\arcsin 4c} + \arcsin 4c \right) \\
&= \frac{2\pi b}{c} \left(\frac{1}{2} \sin(2\arcsin 4c) + \arcsin 4c \right) \\
&= \frac{2\pi b}{c} [\sin(\arcsin 4c)\cos(\arcsin 4c) + \arcsin 4c] \\
&= \frac{2\pi b}{c} [4c\sqrt{1 - \sin^2(\arcsin 4c)} + \arcsin 4c] \\
&= \frac{2\pi b}{c} (4c\sqrt{1 - 16c^2} + \arcsin 4c)
\end{aligned}$$

When $a = 4, b = 2$, then $c = [(a^2 - b^2)/a^4]^{1/2} = \sqrt{3}/8$, the surface area of the ellipsoid is

$$\begin{aligned}
\frac{2\pi b}{c}(4c\sqrt{1 - 16c^2} + \arcsin 4c) &= \frac{32\pi}{\sqrt{3}} \left(\frac{\sqrt{3}}{2} \sqrt{1 - \frac{3}{4}} + \arcsin \frac{\sqrt{3}}{2} \right) \\
&= \frac{32\pi}{\sqrt{3}} \left(\frac{\sqrt{3}}{4} + \frac{\pi}{3} \right) \\
&= 8\pi + \frac{32\sqrt{3}\pi^2}{9} \\
&\approx 85.91374.
\end{aligned}$$

$\square$

8. (15 points) *Nonlinear Springs* - Since Hooke's law is only an approximation to the spring force when the size of the displacement is small, we can increase the accuracy of our model by increasing its complexity. Let's think about a spring whose restoring force is given by

$$F_s(x) = 16x - 0.1x^3$$

for $-6 \leq x \leq 6$, where x is the displacement of the spring from equilibrium.

(a) How much work is done in stretching the spring from its equilibrium position ($x = 0$) to $x = 4.5$?

(b) Compare the result for the previous part with the result that you would get if we used Hooke's law, i.e., $F_h(x) = 16x$.

SOLUTION. (a) The work is required to stretch the spring from its equilibrium position to $x = 4.5$ is

$$\begin{aligned} W &= \int_0^{4.5} F_s x \, dx \\ &= \int_0^{4.5} 16x - 0.1x^3 \, dx \\ &= 8x^2 - \frac{x^4}{40}\bigg|_0^{4.5} \\ &= 162 - \frac{81^2}{640} \end{aligned}$$

(b) If we use Hooke's law, then the work becomes

$$\begin{aligned} W &= \int_0^{4.5} F_h x \, dx \\ &= \int_0^{4.5} 16x \, dx \\ &= 8x^2\bigg|_0^{4.5} \\ &= 162. \end{aligned}$$

□

Topics for Midterm Exam 1:

- Sigma Notation;
- Left/midpoint/right Riemann Sum;
- Definite Integral;
- Fundamental Theorem of Calculus;
- Integrating with Even and Odd Functions;
- Mean Value Theorem for Integrals;
- Substitution Rule for Indefinite/Definite Integrals;
- Area of the Region Between Curves (Integrating with respect to x or y);
- Volume of the Solid by Slicing (Disk/Washer Method) and by Shell Method;
- Length of Curves;
- Surface Area;
- Physical Applications: Mass of a One-Dimensional Object, Work (Hooke's law).