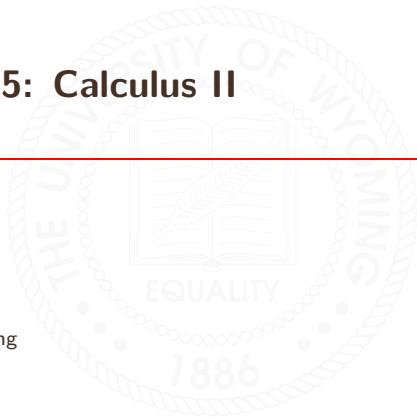


MATH 2205: Calculus II

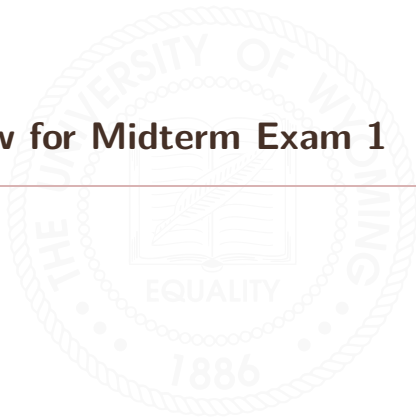
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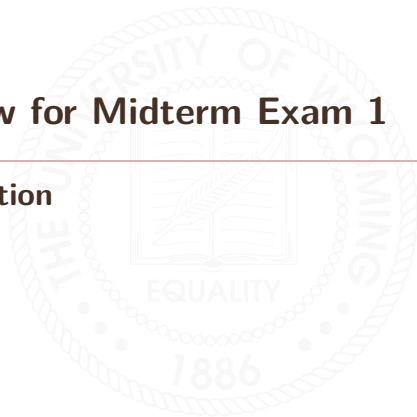


Review for Midterm Exam 1



Review for Midterm Exam 1

Integration



Sigma (Summation) Notation

Proposition

(a) *Constant Multiple Rule.*

$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k.$$

(b) *Addition Rule.*

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k.$$

Sigma (Summation) Notation

Theorem (Sums of Powers of Integers)

Let n be a positive integer and c a real number.

$$\sum_{k=1}^n c = cn,$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

Approximating Areas under Curves

Definition (Riemann Sum)

Suppose f is a function defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx . If x_k^* is any point in the k th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \dots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^n f(x_k^*)\Delta x.$$

is called a *Riemann sum* for f on $[a, b]$. This sum is called

- (a) a *left Riemann sum* if x_k^* is the left endpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = x_{k-1} = a + (k-1)\Delta x$;
- (b) a *right Riemann sum* if x_k^* is the right endpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = x_k = a + k\Delta x$; and
- (c) a *midpoint Riemann sum* if x_k^* is the midpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = (x_{k-1} + x_k)/2 = a + (k-1/2)\Delta x$, for $k = 1, 2, \dots, n$.



Definition (Net Area)

Consider the region R bounded by the graph of a continuous function f and the x -axis between $x = a$ and $x = b$. The *net area* of R is the sum of the areas of the parts of R that lie above the x -axis *minus* the sum of the areas of the parts of R that lie below the x -axis on $[a, b]$.

Definition (Definite Integral)

A function f defined on $[a, b]$ is *integrable* on $[a, b]$ if

$$\lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the *definite integral* of f from a to b , which we write

$$\int_a^b f(x) dx = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k.$$

Definite Integrals

To simplify the calculation, we use equally spaced grid points and right Riemann sums. That is, for each value of n , we let

$\Delta x_k = \Delta x = (b - a)/n$ and $x_k^* = a + k\Delta x$, for $k = 1, 2, \dots, n$.

Then $n \rightarrow \infty$ and $\Delta \rightarrow 0$,

$$\int_a^b f(x) dx = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(a + k\Delta x) \Delta x.$$

Definition (Integrable Functions)

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.



Definition (Reversing Limits and Identical Limits of Integration)

Suppose f is integrable on $[a, b]$.

$$(a) \int_b^a f(x) dx = - \int_a^b f(x) dx.$$

$$(b) \int_a^a f(x) dx = 0.$$



Proposition (Properties of Definite Integrals)

(a) *Integral of a Sum.*

$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

(b) *Constants in Integrals.*

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

(c) *Integrals over Subintervals.*

$$\int_a^b f(x) dx = \int_a^p f(x) dx + \int_p^b f(x) dx.$$



Definition (Area Function)

Let f be a continuous function, for $t \geq a$. The *area function* for f with left endpoint a is

$$A(x) = \int_a^x f(t) dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t -axis on the interval $[a, x]$.



Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus (FTOC), Part I)

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t) dt, \text{ for } a \leq x \leq b.$$

is continuous on $[a, b]$ and differentiable on (a, b) . The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.



Theorem (Fundamental Theorem of Calculus (FTOC), Part II)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b.$$



Proposition (Antiderivatives)

$$(a) \int x^p dx = \frac{x^{p+1}}{p+1} + C, \text{ where } p \neq -1.$$

$$(b) \int x^{-1} dx = \ln |x| + C.$$

$$(c) \int e^x dx = e^x + C.$$

$$(d) \int \sin x dx = -\cos x + C.$$

$$(e) \int \cos x dx = \sin x + C.$$



Proposition (Antiderivatives)

$$(a) \int \frac{1}{1+x^2} dx = \arctan x + C.$$

$$(b) \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C.$$

$$(c) \int \sec x \tan x dx = \sec x + C.$$

$$(d) \int \sec^2 x dx = \tan x + C.$$



Theorem (Integrals of Even and Odd Functions)

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

(a) If f is even, i.e., $f(-x) = f(x)$, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

(b) If f is odd, i.e., $f(-x) = -f(x)$, then $\int_{-a}^a f(x) dx = 0$.



Definition (Average Value of a Function)

The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$



Theorem (Mean Value Theorem for Integrals)

Let f continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(t) dt.$$

Theorem (Substitution Rule for Indefinite Integrals)

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g . On that interval,

$$\int f(g(x))g'(x) dx = \int f(u) du.$$



Substitution Rule

Procedure (Substitution Rule (Change of Variables))

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x) dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u .
4. Write the result in terms of x using $u = g(x)$.



Theorem (Substitution Rule for Definite Integrals)

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g . Then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$



Proposition (Properties of Trig Functions)

$$(a) \sin^2 \theta + \cos^2 \theta = 1.$$

$$(b) \sin 2\theta = 2 \sin \theta \cos \theta.$$

$$(c) \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1.$$

$$(d) \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

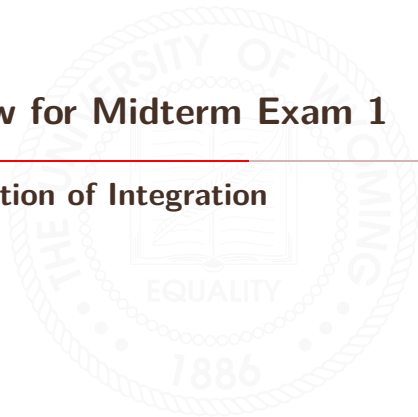
$$(e) \cos^2 \theta = \frac{1 + \cos 2\theta}{2}.$$

$$(f) \sin^2 \theta = \frac{1 - \cos 2\theta}{2}.$$



Review for Midterm Exam 1

Application of Integration



Regions Between Curves

Definition (Approximation of Area of a Region Between Curves)

Suppose that f and g continuous on an interval $[a, b]$ on which $f(x) \geq g(x)$. Partition the interval $[a, b]$ into n subintervals using uniformly spaced grid points separated by a distance $\Delta x = (b - a)/n$. Then the area A of the region bounded by the two curves and the vertical lines $x = a$ and $x = b$ can be approximated by:

$$A \approx \sum_{k=1}^n [f(x_k^*) - g(x_k^*)] \Delta x,$$

where $x_k^* \in [x_{k-1}, x_k]$, $k = 1, 2, \dots, n$, $x_0 = a$, $x_n = b$.

Definition (Area of a Region Between Two Curves)

Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)] dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n [f(x_k^*) - g(x_k^*)] \Delta x.$$

Definition (Area of a Region Between Two Curves with Respect to y)

Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d [f(y) - g(y)] dy.$$



Definition (General Slicing Method)

Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x -axis has an area given by a function A that is integrable on $[a, b]$. Then volume of the solid is

$$V = \int_a^b A(x) dx.$$

Definition (Disk Method about the x -Axis)

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f , the x -axis, and the lines $x = a$ and $x = b$ is revolved about the x -axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2 dx.$$

Definition (Washer Method about the x -Axis)

Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x -axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2] dx.$$



Definition (Disk and Washer Methods about the y -Axis)

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y -axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi [p(y)^2 - q(y)^2] dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2 dy.$$



Definition

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y -axis is

$$V = \int_a^b 2\pi x [f(x) - g(x)] dx.$$



Definition (Arc Length for $y = f(x)$)

Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$



Definition (Arc Length for $x = g(y)$)

Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. Then length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2} dy.$$



Definition (Area of a Surface of Revolution)

Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x -axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx.$$



Definition (Mass of a One-Dimensional Object)

Suppose a thin bar or wire is represented by the interval $a \leq x \leq b$ with a density function ρ (with units of mass per length). The *mass* of the object is

$$m = \int_a^b \rho(x) dx.$$



Definition (Work)

The work done by a variable force F moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x) dx.$$



Theorem (Hooke's law)

The force required to keep the spring in a compressed or stretched position x units from the equilibrium position is

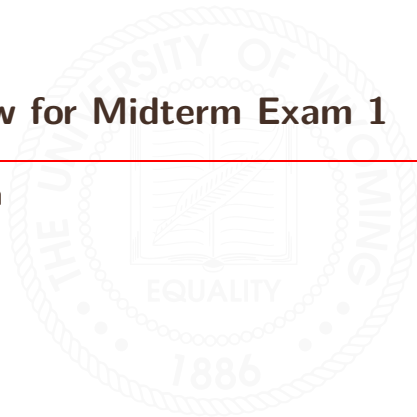
$$F(x) = kx,$$

where the positive spring constant k measures the stiffness of the spring. Note that to stretch the spring to a position $x > 0$, a force $F > 0$ (in the positive position) is required. To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required.



Review for Midterm Exam 1

Algebra



Exponents and Radicals

$$(a) \frac{1}{x^a} = x^{-a}.$$

$$(b) \sqrt[n]{x} = x^{1/n}.$$

$$(c) x^{a+b} = x^a x^b.$$

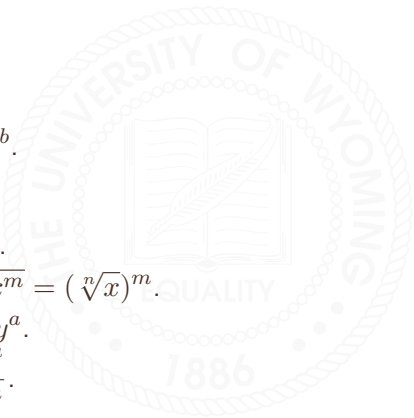
$$(d) x^{a-b} = \frac{x^a}{x^b}.$$

$$(e) x^{ab} = (x^a)^b.$$

$$(f) x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m.$$

$$(g) (xy)^a = x^a y^a.$$

$$(h) \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}.$$



Logarithm

$$(a) y = a^x \implies x = \log_a y.$$

$$(b) \log_e x = \ln x.$$

$$(c) \log_b(xy) = \log_b x + \log_b y.$$

$$(d) \log_b \frac{x}{y} = \log_b x - \log_b y.$$

$$(e) \log_b(x^p) = p \log_b x.$$

$$(f) \log_b(x^{1/p}) = \frac{1}{p} \log_b x.$$

$$(g) \log_b x = \frac{\log_k x}{\log_k b}.$$



Factoring Formulas

$$(a) \quad a^2 - b^2 = (a - b)(a + b).$$

$$(b) \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

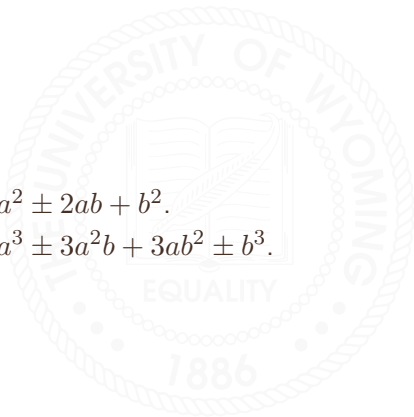
$$(c) \quad a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1}).$$



Binomials

$$(a) \quad (a \pm b)^2 = a^2 \pm 2ab + b^2.$$

$$(b) \quad (a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$



Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

