Full Name: ______________________________ **W-Number:** ____________________

MATH 2205: Calculus II – Midterm Exam 1 Solution

Summer 2019 – Friday, June 07, 2019

Instructions:

- Show all your work and use the space provided on the exam. Correct mathematical notation is required and all partial credit is at discretion of the grader.
- Write neatly and make sure your work is organized.
- Make certain that you have written your Full Name and W-Number in the spaces provided at the top of the exam. Failure to do so may result in a loss of points.
- No aids beyond a scientific, non-graphing calculator are allowed. This means no notes, no cell phones, etc., are permitted during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 9 pages. Please check to see that your copy has all the pages.

For Instructor Use Only

Question	1	2	3	4	5	6	7	8	Total
Points	15	20	10	10	10	10	10	15	100
Mark									

Formulae you may find useful:

- $\displaystyle\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$
- $\displaystyle\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.
- $\displaystyle\int x^{-1} \, dx = \ln|x| + C.$
- $\displaystyle\int e^x \, dx = e^x + C.$
- $\displaystyle\int \sin x \, dx = -\cos x + C.$
- $\displaystyle\int \cos x \, dx = \sin x + C.$
- $\displaystyle\int \frac{1}{1+x^2} \, dx = \arctan x + C.$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C.$
- $\displaystyle\int \sec x \tan x \, dx = \sec x + C.$
- $\displaystyle\int \sec^2 x \, dx = \tan x + C.$
- $\cos^2 \theta = \dfrac{1 + \cos 2\theta}{2}.$
- $\sin^2 \theta = \dfrac{1 - \cos 2\theta}{2}.$
- $\sin^2 \theta + \cos^2 \theta = 1.$
- $\sin 2\theta = 2 \sin \theta \cos \theta.$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$

1. (15 points) Circle TRUE if the statement is true or FALSE if it is not, and justify your choice briefly.

   (a) TRUE or $\boxed{\text{FALSE}}$: Suppose that f is defined on the interval $[a, b]$ which is partitioned to n subintervals of equal length. The right Riemann sum for f on $[a, b]$ is always greater than the left Riemann sum for f on $[a, b]$.
   When f is constant function, left Riemann sum equals right Riemann sum.

   (b) TRUE or $\boxed{\text{FALSE}}$: The piecewise function $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ -3x+6 & \text{if } 1 < x \le 2 \end{cases}$ is *not* integrable on $[0, 2]$.
   f is bounded function with only 1 discontinuity.

   (c) $\boxed{\text{TRUE}}$ or FALSE: $\int_0^1 e^{x+\ln 2}\,dx = 2\int_0^1 e^x\,dx$.
   $\int_0^1 e^x e^{\ln 2}\,dx = \int_0^1 2e^x\,dx = 2\int_0^1 e^x\,dx$.

   (d) $\boxed{\text{TRUE}}$ or FALSE: If the function f is always nonnegative on the interval $[a, b]$, then the area and the net area between the curve and the x-axis from a to b are equal.
   Yes! The net area is the area above the x-axis minus the area below x-axis, now the area below the x-axis is 0.

   (e) $\boxed{\text{TRUE}}$ or FALSE: $\sum_{k=-2}^{1000000} (k+2)(k+1)k = \sum_{k=1}^{1000000} (k+2)(k+1)k$.

$$\begin{aligned}
\sum_{k=-2}^{1000000} (k+2)(k+1)k &= \sum_{k=-2}^{0} (k+2)(k+1)k + \sum_{k=1}^{1000000} (k+2)(k+1)k. \\
&= 0 + \sum_{k=1}^{1000000} (k+2)(k+1)k. \\
&= \sum_{k=1}^{1000000} (k+2)(k+1)k.
\end{aligned}$$

2. (20 points) Evaluate the following integrals.

(a) $\displaystyle\sum_{k=1}^{20}(3k+2)^2.$

Solution.

$$\begin{aligned}
\sum_{k=1}^{20}(3k+2)^2 &= \sum_{k=1}^{20}(9k^2+12k+4) \\
&= 9\sum_{k=1}^{20}k^2 + 12\sum_{k=1}^{20}k + 4\sum_{k=1}^{20}1 \\
&= 9\left.\frac{n(n+1)(2n+1)}{6}\right|_{n=20} + 12\left.\frac{n(n+1)}{2}\right|_{n=20} + 4\times 20 \\
&= 9\frac{20\times 21\times 41}{6} + 12\frac{20\times 21}{2} + 80 \\
&= 28430.
\end{aligned}$$

□

(b) $\displaystyle\sum_{k=11}^{50}3(2k+1).$

Solution.

$$\begin{aligned}
\sum_{k=11}^{50}3(2k+1) &= \sum_{k=11}^{50}(6k+3) \\
&= \sum_{k=1}^{50}(6k+3) - \sum_{k=1}^{10}(6k+3) \\
&= 6\sum_{k=1}^{50}k + 3\sum_{k=1}^{50}1 - 6\sum_{k=1}^{10}k - 3\sum_{k=1}^{10}1 \\
&= 6\left.\frac{n(n+1)}{2}\right|_{n=50} + 3\times 50 - 6\left.\frac{n(n+1)}{2}\right|_{n=10} - 3\times 10 \\
&= 6\frac{50\times 51}{2} + 3\times 50 - 6\frac{10\times 11}{2} - 3\times 10 \\
&= 7440.
\end{aligned}$$

□

(c) $\displaystyle\int_{-1}^{3} x\sin(x^2+2)\,dx.$

SOLUTION.

$$\begin{aligned}\int_{-1}^{3} x\sin(x^2+2)\,dx &= \frac{1}{2}\int_{-1}^{3}\sin(x^2+2)\,2xdx\\ &= \frac{1}{2}\int_{u(-1)}^{u(3)}\sin(u)\,du\\ &= \frac{1}{2}\int_{3}^{11}\sin(u)\,du\\ &= -\frac{1}{2}\cos u\Big|_{3}^{11}\\ &= -\frac{1}{2}(\cos 11-\cos 3)\\ &\approx -0.49721.\end{aligned}$$

□

(d) $\displaystyle\int_{-\pi/2}^{\pi/2} x(e^x+e^{-x})+2\cos^2(x)\,dx.$

SOLUTION. Observe that $x(e^x+e^{-x})$ is odd, and $2\cos^2(x)$ is even on $[-\frac{\pi}{2},\frac{\pi}{2}]$.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} x(e^x+e^{-x})+2\cos^2(x)\,dx &= \int_{-\pi/2}^{\pi/2} x(e^x+e^{-x})\,dx+\int_{-\pi/2}^{\pi/2} 2\cos^2(x)\,dx\\ &= 0+2\int_{0}^{\pi/2}2\cos^2(x)\,dx\\ &= 2\int_{0}^{\pi/2}\cos(2x)+1\,dx\\ &= 2\left(\int_{0}^{\pi/2}\cos(2x)\,dx+\int_{0}^{\pi/2}1\,dx\right)\\ &= 2\left(\frac{1}{2}\int_{0}^{\pi/2}\cos(2x)2\,dx+\pi/2\right)\\ &= 2\left(\frac{1}{2}\int_{u(0)}^{u(\pi/2)}\cos(u)\,du+\pi/2\right)\\ &= 2\left(\frac{1}{2}\sin u\Big|_{0}^{\pi}+\pi/2\right)\\ &= \pi.\end{aligned}$$

□

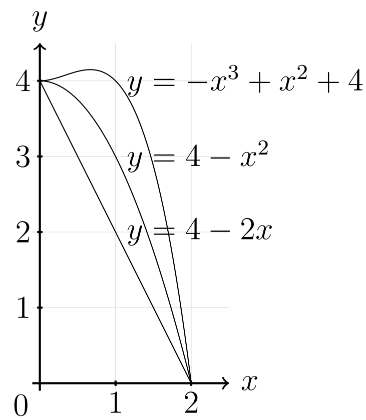
3. (10 points) Use right Riemann Sum to approximate the net area of the region bounded by the graph of $f(x) = (x+4)(x-4)x$ and x-axis on $[0, 20]$ for $n = 10$.

SOLUTION. The interval $[0, 20]$ is partitioned to $n = 10$ subintervals of the same length Δx, where $\Delta x = \frac{b-a}{n} = \frac{20-0}{10} = 2$. We are using right Riemann sum, so we let $x_k^* = x_k = x_0 + k\Delta x = 2k$. Then the right Riemann sum is

$$\begin{aligned}
\sum_{k=1}^{10} f(x_k^*)\Delta x &= \sum_{k=1}^{10} f(x_k)\Delta x \\
&= \sum_{k=1}^{10} f(2k) \cdot 2 \\
&= 2\sum_{k=1}^{10} (2k+4)(2k-4)2k \\
&= 2\sum_{k=1}^{10} (4k^2-16)2k \\
&= 2\sum_{k=1}^{10} (8k^3-32k) \\
&= 2\left(\sum_{k=1}^{10} 8k^3 - \sum_{k=1}^{10} 32k\right) \\
&= 2\left(8\sum_{k=1}^{10} k^3 - 32\sum_{k=1}^{10} k\right) \\
&= 2\left(8\left.\frac{n^2(n+1)^2}{4}\right|_{n=10} - 32\left.\frac{n(n+1)}{2}\right|_{n=10}\right) \\
&= 2\left(8\frac{100 \times 121}{4} - 32\frac{10 \times 11}{2}\right) \\
&= 44880.
\end{aligned}$$

$\square$

4. (10 points) Here is a picture containing the graphs of three functions. Which is larger,



the area between the curves $y = -x^3 + x^2 + 4$ and $y = 4 - x^2$, or the area between the curves $y = 4 - x^2$ and $y = 4 - 2x$?

Solution. Let $f(x) = -x^3 + x^2 + 4$, $g(x) = 4 - x^2$, and $h(x) = 4 - 2x$. The area between $y = -x^3 + x^2 + 4$ and $y = 4 - x^2$ is

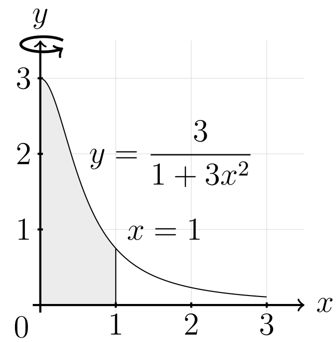
$$\begin{aligned}
\int_a^b f(x) - g(x)\,dx &= \int_0^2 (-x^3 + x^2 + 4) - (4 - x^2)\,dx \\
&= \int_0^2 -x^3 + 2x^2\,dx \\
&= -\frac{x^4}{4} + \frac{2x^3}{3}\Big|_0^2 \\
&= -4 + \frac{16}{3} \\
&= \frac{4}{3}.
\end{aligned}$$

And the area between $y = 4 - x^2$ and $y = 4 - 2x$ is

$$\begin{aligned}
\int_a^b g(x) - h(x)\,dx &= \int_0^2 (4 - x^2) - (4 - 2x)\,dx \\
&= \int_0^2 -x^2 + 2x\,dx \\
&= -\frac{x^3}{3} + x^2\Big|_0^2 \\
&= -\frac{8}{3} + 4 \\
&= \frac{4}{3}.
\end{aligned}$$

The area between $y = -x^3 + x^2 + 4$ and $y = 4 - x^2$ is same as the area between $y = 4 - x^2$ and $y = 4 - 2x$. □

5. (10 points) Let R be the region bounded by $y = \dfrac{3}{1+3x^2}$, $x = 1$, x-axis and y-axis. Indicate (by shading) the region R in the graph below. Find the volume of the solid generated when R is revolved about the y-axis.



SOLUTION. Note that both Disk Method and Shells Method is applicable to this problem. However, if we use Disk Method, we need to split the region into two subregions. So Shells Method is preferred. Therefore, the interval we integrate on is $[0, 1]$, then

$$\begin{aligned}
\int_0^1 2\pi x f(x)\, dx &= \int_0^1 2\pi x \frac{3}{1+3x^2}\, dx \\
&= \pi \int_0^1 \frac{1}{\underbrace{1+3x^2}_{u}} \underbrace{6x\, dx}_{du} \\
&= \pi \int_{u(0)}^{u(1)} \frac{1}{u}\, du \\
&= \pi \ln |u| \Big|_{u(0)=1}^{u(1)=4} \\
&= \pi \ln 4 \\
&\approx 0.69315.
\end{aligned}$$

□

6. (10 points) Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ on $[-\ln 3, \ln 3]$ by integrating with respect to x.

SOLUTION. Let $f(x) = \frac{1}{2}(e^x + e^{-x})$, then

$$f'(x) = \frac{1}{2}(e^x - e^{-x}) \implies f'(x)^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x}).$$

Hence we can apply the formula to calculate the arc length as follows,

$$\begin{aligned}
\int_{-\ln 3}^{\ln 3} \sqrt{1 + f'(x)^2}\,dx &= \int_{-\ln 3}^{\ln 3} \left[1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})\right]^{1/2} dx \\
&= \int_{-\ln 3}^{\ln 3} \left[\frac{4}{4} + \frac{1}{4}(e^{2x} - 2 + e^{-2x})\right]^{1/2} dx \\
&= \int_{-\ln 3}^{\ln 3} \left[\frac{1}{4}(e^{2x} - 2 + 4 + e^{-2x})\right]^{1/2} dx \\
&= \frac{1}{2}\int_{-\ln 3}^{\ln 3} \left[(e^{2x} + 2 + e^{-2x})\right]^{1/2} dx \\
&= \frac{1}{2}\int_{-\ln 3}^{\ln 3} \left[(e^{x} + e^{-x})^2\right]^{1/2} dx \\
&= \frac{1}{2}\int_{-\ln 3}^{\ln 3} (e^{x} + e^{-x})\,dx \\
&= \frac{1}{2}(e^x - e^{-x})\Big|_{-\ln 3}^{\ln 3} \\
&= \frac{1}{2}[(e^{\ln 3} - e^{-\ln 3}) - (e^{-\ln 3} - e^{\ln 3})] \\
&= \frac{1}{2}[2(e^{\ln 3} - e^{-\ln 3})] \\
&= e^{\ln 3} - e^{-\ln 3} \\
&= 3 - \frac{1}{3} \\
&= \frac{8}{3}.
\end{aligned}$$

□

7. (10 points) The graph of $f(x) = 5\sqrt{x}$ on the interval $[1, 3]$ is revolved about the x-axis. What is the area of the surface generated?

Solution. Provided that $f(x) = 5\sqrt{x} = 5x^{1/2}$, then

$$f'(x) = 5 \times \frac{1}{2}x^{1/2-1} = \frac{5}{2}x^{-1/2} \implies f'(x)^2 = \frac{5^2}{2^2}(x^{-1/2})^2 = \frac{25}{4}x^{-1}.$$

Then we can apply the formula to obtain the surface area as follows,

$$\begin{aligned}
A &= \int_1^3 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \\
&= \int_1^3 2\pi \times 5x^{1/2}\left(1 + \frac{25}{4}x^{-1}\right)^{1/2} dx \\
&= 10\pi \int_1^3 \left[x\left(1 + \frac{25}{4}x^{-1}\right)\right]^{1/2} dx \\
&= 10\pi \int_1^3 \left(\underbrace{x + \frac{25}{4}}_{u}\right)^{1/2} \underbrace{dx}_{du} \\
&= 10\pi \int_{u(1)}^{u(3)} u^{1/2}\, du \\
&= 10\pi \frac{2u^{3/2}}{3}\Bigg|_{u(1)=29/4}^{u(3)=37/4} \\
&= \frac{20\pi}{3}u^{3/2}\Bigg|_{29/4}^{37/4} \\
&= \frac{20\pi}{3}\left[\left(\frac{37}{4}\right)^{3/2} - \left(\frac{29}{4}\right)^{3/2}\right] \\
&\approx 180.35997.
\end{aligned}$$

□

8. (15 points) *Physical Applications.*

   (a) Find the mass of the thin bar with the given density function $\rho(x) = \dfrac{4}{1+x^2}$ for $0 \le x \le 1$.

   (b) A spring, which obeys Hooke's law, on a horizontal surface can be stretched and held 0.4 m from its equilibrium position with a force of 40 N. Is the work done in stretching the spring 0.25 m from its equilibrium position equal to the work done in stretching the spring 0.35 m if it has already been stretched 0.1 from its equilibrium position?

SOLUTION. (a) The mass of the thin bar is

$$\begin{aligned} m &= \int_0^1 \rho(x)\,dx \\ &= \int_0^1 \frac{4}{1+x^2}\,dx \\ &= 4\int_0^1 \frac{1}{1+x^2}\,dx \\ &= 4\arctan x\Big|_0^1 \\ &= 4(\arctan 1 - \arctan 0) \\ &= 4\left(\frac{\pi}{4} - 0\right) \\ &= \pi. \end{aligned}$$

(b) By Hooke's law, we can find the spring constant as follows,

$$F(x) = kx \implies k = \frac{F(x)}{x} = \frac{40}{0.4} = 100.$$

The work done in stretching the spring 0.25 m from its equilibrium position is

$$W_1 = \int_0^{0.25=1/4} F(x)\,dx = \int_0^{1/4} 100x\,dx = 50x^2\Big|_0^{1/4} = 50\left[\left(\frac{1}{4}\right)^2 - 0^2\right] = \frac{25}{8}.$$

While the work done in stretching the spring 0.35 m from 0.1 m is

$$W_2 = \int_{0.1=1/10}^{0.35=7/20} F(x)\,dx = \int_{1/10}^{7/20} 100x\,dx = 50x^2\Big|_{1/10}^{7/20} = 50\left[\left(\frac{7}{20}\right)^2 - \left(\frac{1}{10}\right)^2\right] = \frac{45}{8}.$$

It is shown that $W_2 = \dfrac{45}{8} > W_1 = \dfrac{25}{8}$, then two works are not equal.

$\square$