MATH 2205 - Calculus II Lecture Notes 25

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1 Sequences and Infinite Series

1.1 Sequences

Definition 1.1 (Sequence)**.** A *sequence* $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a *recurrence relation* of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an *explicit formula* of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Definition 1.2 (Limit of a Sequence)**.** If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases – that is, if a_n can be made arbitrarily close to L by taking n sufficiently large – then we say $\lim\limits_{n\to\infty} a_n = L$ exists, and the sequence *converges* to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence *diverges*.

Theorem 1.1 (Limits of Sequences from Limits of Functions)**.** Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim\limits_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

Theorem 1.2 (Limit Laws for Sequences)**.** Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

(a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$.
(b) $\lim\limits_{n\to\infty} ca_n = cA$, where c is a real number.
(c) $\lim\limits_{n\to\infty} a_n b_n = AB$.
(d) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ provided $B \neq 0$.

Definition 1.3 (Terminology for Sequences)**.**

(a) $\{a_n\}$ is *increasing* if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$.
(b) $\{a_n\}$ is *nondecreasing* if $a_{n+1} \geq a_n$; for example, $\{0, 1, 1, 1, 2, 2, 3, \ldots\}$.
(c) $\{a_n\}$ is *decreasing* if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -2, \ldots\}$.
(d) $\{a_n\}$ is *nonincreasing* if $a_{n+1} \leq a_n$; for example, $\{2, 1, 1, 0, -2, -2, -3, \ldots\}$.
(e) $\{a_n\}$ is *monotonic* if it is either nonincreasing or nondecreasing (it moves in one direction).
(f) $\{a_n\}$ is *bounded* if there is number M such that $|a_n| \leq M$, for all relevant values of n.

Theorem 1.3 (Squeeze Theorem for Sequences)**.** Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Theorem 1.4 (Bounded Monotonic Sequences)**.** A bounded monotonic sequence converges.

Theorem 1.5 (Growth Rates of Sequences)**.** The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s and $b > 1$.

1.2 Limits of Functions

Theorem 1.6 (Limits of Linear Functions)**.** Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x\to a} f(x) = f(a) = ma + b.$$

Theorem 1.7 (Limit Laws)**.** Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where c is a real number, and $m > 0$ and $n > 0$ are integers.

(a) *Sum*: $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
(b) *Difference*: $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
(c) *Constant multiple*: $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$.
(d) *Product*: $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$.
(e) *Quotient*: $\displaystyle\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
(f) *Power*: $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$.

Theorem 1.8 (Limits of Polynomial and Rational Functions)**.** Assume p and q are polynomials and a is a constant.

(a) Polynomial functions: $\lim_{x\to a} p(x) = p(a)$.
(b) Rational functions: $\displaystyle\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$.

Theorem 1.9 (The Squeeze Theorem)**.** Assume the function f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Theorem 1.10 (Limits at Infinity of Powers and Polynomials)**.** Let n be a positive integer and let p be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

(a) $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
(b) $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.
(c) $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$.
(d) $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Theorem 1.11 (End Behavior of e^x, e^{-x}, and $\ln x$)**.** The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits (see Figure 1):

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

$$\lim_{x\to0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty.$$

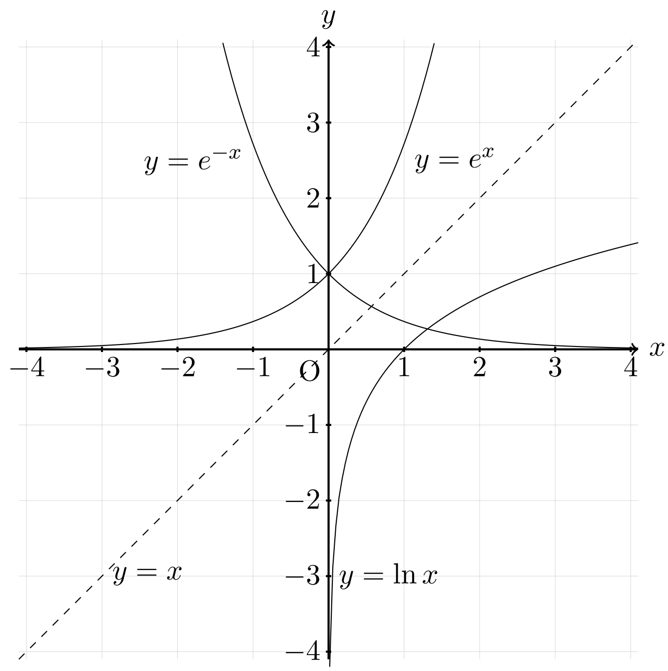

Figure 1: Graphs of e^x, e^{-x}, $\ln x$: $y = e^{-x}$ and $y = e^x$ are symmetric about y-axis, and $y = e^x$ and $y = \ln x$ are symmetric about $y = x$.

Theorem 1.12 (L'Hôpital's Rule)**.** Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is repaced with $x \to \pm\infty$, $x \to a^+$, $x \to a^-$.

(b) If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is repaced with $x \to \pm\infty$, $x \to a^+$, $x \to a^-$.

1.3 Infinite Series

Definition 1.4 (Infinite series)**.** Given a sequence $\{a_1, a_2, a_3, \ldots, \}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an *infinite series.* The *sequence of partial sums* $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series *converges* to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also *diverges.*

1.4 Convergent Series

Theorem 1.13 (Properties of Convergent Series)**.**

(a) Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c \sum a_k = cA$.

(b) Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.

(c) If M is a positive integer, then $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Definition 1.5 (Absolute and Conditional Convergence)**.** If $\sum|a_k|$ converges, then $\sum a_k$ *converges absolutely.* If $\sum|a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ *converges conditionally.*

Theorem 1.14 (Absolute Convergence Implies Convergence)**.** If $\sum|a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum|a_k|$ diverges.

1.5 Harmonic Series, Alternating Harmonic Series, and p-Series

Theorem 1.15 (Harmonic Series)**.** The harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges – even though the terms of the series approach zero.

Proposition 1.1.

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \approx \ln n + \gamma,$$

where $\gamma \approx 0.57721\ldots$.

Theorem 1.16 (Alternating Harmonic Series)**.** The alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges (even though the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges).

Theorem 1.17 (Convergence of the p-Series)**.** The *p-series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

1.6 Geometric Sequences and Geometric Series

Definition 1.6 (Geometric Sequences)**.** A sequence has the form $\{r^n\}$ or $\{ar^n\}$, where the ratio r, a are real numbers, is called a geometric sequence.

Theorem 1.18 (Geometric Sequences)**.** Let r be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1, \\ 1 & \text{if } r = 1, \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.

Theorem 1.19 (Geometric Series)**.** Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges. More generally,

$$\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}.$$

1.7 Power Series

Definition 1.7 (Power Series)**.** A *power series* has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the *coefficients* of the power series and a is the *center* of the power series. The set of values of x for which the series converges is its *interval of convergence*. The *radius of convergence* of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Definition 1.8 (Taylor Polynomials)**.** Let f be a function with $f', f'', \ldots, f^{(n)}$ defined at a. Then nth-order Taylor polynomial for f with its center at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n = f(a), p_n'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} c_k(x-a)^k,$$

where the *coefficients* are

$$c_k = \frac{f^{(k)}(a)}{k!}, \text{for } k = 0, 1, 2, \ldots.$$

Definition 1.9 (Taylor/MacLaurin Series for a Function)**.** Suppose the function f has derivatives of all orders on an interval centered at the point a. The *Taylor series* for f centered at a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a *MacLaurin series*.

Proposition 1.2 (MacLaurin series)**.**

(a) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, for $|x| < \infty$.

(b) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, for $|x| < \infty$.

(c) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, for $|x| < \infty$.

(d) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, for $|x| < 1$.

(e) $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$, for $|x| < 1$.

(f) $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$, for $-1 < x \leq 1$.

(g) $-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$, for $-1 \leq x < 1$.

Theorem 1.20 (Convergence of Power Series)**.** A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

(a) The series converges for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

(b) There is a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is R.

(c) The series converges only at a, in which case the radius of convergence is $R = 0$.

Theorem 1.21 (Combining Power Series)**.** Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

(a) *Sum and difference*: The power series $\sum (c_k \pm d_k) x^k$ converges to $f(x) \pm g(x)$ on I.
(b) *Multiplication by a power*: Suppose m is an integer such that $k + m \geq 0$ for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x \to 0} x^m f(x)$.
(c) *Composition*: If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Theorem 1.22 (Differentiating and Integrating Power Series)**.** Suppose the power series $\sum c_k (x - a)^k$ converges for $|x - a| < R$ and defines a function f on that interval.

(a) Then f is differentiable (which implies continuous) for $|x - a| < R$, and f' is found by differentiating the power series for f term by term: that is,

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum c_k (x-a)^k = \sum \frac{d}{dx} c_k (x-a)^k = \sum k c_k (x-a)^{k-1},$$

for $|x - a| < R$.
(b) The indefinite integral of f is found by integrating the power series for f term by term: that is,

$$\int f(x)\, dx = \int \sum c_k (x-a)^k \, dx = \sum \int c_k (x-a)^k \, dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$

for $|x - a| < R$, where C is an arbitrary constant.

1.8 The Divergence, Integral, Ratio, Root, Comparison, Limit Comparison, Alternating Series Tests

Theorem 1.23 (Contrapositive and Converse)**.** If the statement "if p, then q" (i.e., $p \implies q$) is true, then its *contrapositive*, "if (not q), then (not p)" (i.e., $\neg q \implies \neg p$), is also true. However its *converse*, "if q, then p" (i.e., $q \implies p$), is not necessary true. In short,

$$\begin{aligned} p \implies q &\equiv \neg q \implies \neg p, \\ p \implies q &\not\equiv q \implies p, \end{aligned}$$

where $A \equiv B$ means A and B are equivalent.

Example 1.1. Assume that Laramie is one of the cities in Wyoming, and both Laramie and Wyoming are unique in our universe.
Statement: If I live in Laramie, then I live in Wyoming. (true)
Contrapositive: If I don't live in Wyoming, then I don't live in Laramie. (true)
Converse: If I live in Wyoming, then I live in Laramie. (false)

Theorem 1.24 (Divergence Test)**.** If $\sum a_k$ converges, then $\lim_{k \to \infty} a_k = 0$. Equivalently, if $\lim_{k \to \infty} a_k \neq 0$, then the series diverges.

Note: The converse of the above statement, if $\lim_{k\to\infty} a_k = 0$, then the series converges, might not be true.

Theorem 1.25 (Integral Test)**.** Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

Theorem 1.26 (Ratio Test)**.** Let $\sum a_k$ be an infinite series with positive terms and let $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

(a) If $0 \leq r < 1$, the series converges.
(b) If $r > 1$ (including $r = \infty$), the series diverges.
(c) If $r = 1$, the test is inconclusive.

Theorem 1.27 (Root Test)**.** Let $\sum a_k$ be an infinite series with nonnegative terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.

(a) If $0 \leq \rho < 1$, the series converges.
(b) If $\rho > 1$ (including $\rho = \infty$), the series diverges.
(c) If $\rho = 1$, the test is inconclusive.

Theorem 1.28 (Comparison Test)**.** Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

(a) If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(b) If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Theorem 1.29 (Limit Comparison Test)**.** Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

(a) If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
(b) If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(c) If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Theorem 1.30 (Alternating Series Test)**.** The alternating series $\sum (-1)^{k+1} a_k$ converges provided

(a) the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$), for k greater than some index N) and
(b) $\lim_{k\to\infty} a_k = 0$.

Procedure 1.1 (Guidelines for Choosing a Test)**.**

(a) Begin with Divergence Test. If you show that $\lim_{k\to\infty} a_k \neq 0$, then the series diverges and your work is finished. The order of growth rates of sequences is useful for evaluating $\lim_{k\to\infty} a_k$.
(b) • Geometric series: $\sum ar^k$ converges for $|r| < 1$ and diverges for $|r| \geq 1 (a \neq 0)$.

- p-series: $\sum \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
- Check also for a telescoping series.

(c) If the genearl kth term of the series looks like a function you can integrate, then try the Integral Test.
(d) If the general kth term of the series involves $k!$, k^k, a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.
(e) If the general kth term of the series is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test.
(f) If the sign of the terms is alternating, use the Alternating Series Test.

1.9 Remainder and Approximation of Series

Definition 1.10 (Remainder)**.** The *remainder* is the error in approximating a convergent series by the sum of its first n terms, that is,

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

Theorem 1.31 (Estimating Series with Positive Terms)**.** Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \displaystyle\sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \displaystyle\sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n < \int_n^{\infty} f(x)\, dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\, dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\, dx.$$

Theorem 1.32 (Remainder in Alternating Series)**.** Let $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \leq a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

2 Algebra

2.1 Exponents and Radicals

(a) $\dfrac{1}{x^a} = x^{-a}$.
(b) $\sqrt[n]{x} = x^{1/n}$.
(c) $x^{a+b} = x^a x^b$.
(d) $x^{a-b} = \dfrac{x^a}{x^b}$.

(e) $x^{ab} = (x^a)^b$.
(f) $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.
(g) $(xy)^a = x^a y^a$.
(h) $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$.

2.2 Logarithm

(a) $y = a^x \implies x = \log_a y$.
(b) $\log_e x = \ln x$.
(c) $\log_b(xy) = \log_b x + \log_b y$.
(d) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$.
(e) $\log_b(x^p) = p \log_b x$.
(f) $\log_b(x^{1/p}) = \dfrac{1}{p} \log_b x$.
(g) $\log_b x = \dfrac{\log_k x}{\log_k b}$.

2.3 Factoring Formulas

(a) $a^2 - b^2 = (a-b)(a+b)$.
(b) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
(c) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$.

2.4 Binomials

(a) $(a \pm b)^2 = a^2 \pm 2ab + b^2$.
(b) $(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$.

2.5 Completing the Square

(a) $(x \pm p)^2 = x^2 \pm 2px + p^2$.
(b)

$$\begin{aligned} x^2 \pm bx + c &= x^2 \pm 2\frac{b}{2}x + c \\ &= x^2 \pm 2\frac{b}{2}x + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 \\ &= \left(x \pm \frac{b}{2}\right)^2 + c - \frac{b^2}{4}. \end{aligned}$$

(c)

$$\begin{aligned}
ax^2 \pm bx + c &= a\left(x^2 \pm \frac{b}{a}x\right) + c \\
&= a\left(x^2 \pm 2\frac{b}{2a}x\right) + c \\
&= a\left[x^2 \pm 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right] + c - a\left(\frac{b}{2a}\right)^2 \\
&= a\left(x \pm \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= (\sqrt{a})^2\left(x \pm \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= \left(\sqrt{a}x \pm \frac{\sqrt{a}b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= \left(\sqrt{a}x \pm \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a}.
\end{aligned}$$

2.6 Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$