MATH 2205 - Calculus II Lecture Notes 21

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1 Sequences and Series

1.1 Review of Comparison, Limit Comparison, Alternating Series Tests

Theorem 1.1 (Comparison Test)**.** Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

(a) If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(b) If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Theorem 1.2 (Limit Comparison Test)**.** Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

(a) If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
(b) If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(c) If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Theorem 1.3 (Alternating Series Test)**.** The alternating series $\sum (-1)^{k+1} a_k$ converges provided

(a) the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$), for k greater than some index N) and
(b) $\lim\limits_{k\to\infty} a_k = 0$.

Theorem 1.4 (Alternating Harmonic Series)**.** The alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges (even though the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges).

1.2 Convergent Series

Theorem 1.5 (Properties of Convergent Series)**.**

(a) Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c \sum a_k = cA$.
(b) Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum (a_k \pm b_k)$ converges, and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
(c) If M is a positive integer, then $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Example 1.1 (Using properties of series)**.** Evaluate the infinite series

$$S = \sum_{k=1}^{\infty} \left[5 \left(\frac{2}{3} \right)^k - \frac{2^{k-1}}{7^k} \right].$$

Solution.

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \left[5 \left(\frac{2}{3} \right)^k - \frac{2^{k-1}}{7^k} \right] \\ &= \sum_{k=1}^{\infty} 5 \left(\frac{2}{3} \right)^k - \sum_{k=1}^{\infty} \frac{2^{k-1}}{7^k} \\ &= \sum_{k=1}^{\infty} 5 \left(\frac{2}{3} \right)^k - \sum_{k=1}^{\infty} 2^{-1} \left(\frac{2}{7} \right)^k \\ &= \frac{5 \cdot \left(\frac{2}{3}\right)^1}{1 - \frac{2}{3}} - \frac{\frac{1}{2}\left(\frac{2}{7}\right)^1}{1 - \frac{2}{7}} \qquad [\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}] \\ &= 10 - \frac{1}{5} \\ &= \frac{49}{5}. \end{aligned}$$

□

Definition 1.1 (Absolute and Conditional Convergence)**.** If $\sum |a_k|$ converges, then $\sum a_k$ *converges absolutely*. If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ *converges conditionally.*

Theorem 1.6 (Absolute Convergence Implies Convergence)**.** If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Example 1.2 (Absolute and conditional convergence)**.** Determine whether the following series diverge, converge absolutely, or converge conditionally.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^3}}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+1}$.

Solution.

(a) Identify that $a_k = \dfrac{(-1)^{k+1}}{\sqrt{k}}$, then test the absolute convergence by considering the convergence of $\sum_{k=1}^{\infty} |a_k|$,

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}},$$

which is a p-series with $p = 1/2$, therefore, $\sum_{k=1}^{\infty} |a_k|$ diverges. Note that $\sum_{k=1}^{\infty} a_k$ is an alternating series. Then we perform Alternating Series Test, noting that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$, of which $b_k = \frac{1}{\sqrt{k}}$ is decreasing and $\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$. Hence $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges. By definition, the given series converges conditionally.

(b) Similarly, let $a_k = \dfrac{(-1)^{k+1}}{\sqrt[3]{k}}$, and

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{k^3}} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

which is a convergent p-series for $p = 3/2 > 1$. Therefore, $\sum_{k=1}^{\infty} |a_k|$ converges, and thus $\sum_{k=1}^{\infty} a_k$ converges absolutely.

(c) In this series, $a_k = \dfrac{\sin k}{k^2}$ and consider $|a_k| = \left|\frac{\sin k}{k^2}\right|$. It is known that $|\sin k| \leq 1$, then dividing both sides by k^2 gives

$$|a_k| = \left| \frac{\sin k}{k^2} \right| = \frac{|\sin k|}{k^2} \leq \frac{1}{k^2}.$$

Let $b_k = \frac{1}{k^2}$, and $a_k \leq b_k$ for $k = 1, 2, 3, \ldots$. We see that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$, which is convergent. Therefore, by Comparison Test, $\sum_{k=1}^{\infty} |a_k|$ converges, which forces $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges. In other words, the series converges absolutely.

(d) Let $a_k = \frac{(-1)^k k}{k+1}$, and

$$\begin{aligned}
\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \frac{(-1)^k k}{k+1} \\
&= [\lim_{k\to\infty} (-1)^k] \cdot \left(\lim_{k\to\infty} \frac{k}{k+1} \right) \\
&= [\lim_{k\to\infty} (-1)^k] \cdot \left(\lim_{k\to\infty} \frac{k/k}{k/k + 1/k} \right) \\
&= [\lim_{k\to\infty} (-1)^k] \cdot \left(\lim_{k\to\infty} \frac{1}{1 + 1/k} \right) \\
&= [\lim_{k\to\infty} (-1)^k] \cdot 1 \\
&= \lim_{k\to\infty} (-1)^k,
\end{aligned}$$

which does not exist. By Divergence Test, $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k k}{k+1}$ diverges, and by Theorem 1.6, $\sum_{k}^{\infty} |a_k| = \sum_{k}^{\infty} \frac{k}{k+1}$ also diverges.

□

1.3 Properties of Power Series

Definition 1.2 (Power Series)**.** A *power series* has the general form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the *coefficients* of the power series and a is the *center* of the power series. The set of values of x for which the series converges is its *interval of convergence*. The *radius of convergence* of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Example 1.3 (Interval and radius of convergences)**.** Find the interval and radius of convergence for each power series.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(x-2)^k}{4^k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} k!\, x^k$.

Solution.

(a) The center of the power series is 0 and the terms of the series are $x^k/k!$. Due to the precence of the factor $k!$, we test the series for absolute convergence using the Ratio Test:

$$r = \lim_{k\to\infty} \frac{|x^{k+1}/(k+1)!\,|}{|x^k/k!\,|} = \lim_{k\to\infty} \frac{|x^{k+1}|}{|x^k|}\frac{k!}{(k+1)!} = \lim_{k\to\infty} |x| \frac{k!}{k!\cdot(k+1)} = |x| \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

Notice that in taking the limit as $k \to \infty$, x is held fixed. Because $r = 0$ for all real numbers x, the series converges absolutely for all x. By Theorem 1.6, we conclude that the series converges for all x. Therefore, the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

(b) We test for absolute convergence using the Root Test:

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k(x-2)^k}{4^k}\right|} = \lim_{k\to\infty} \left[\left(\frac{|x-2|}{4}\right)^k\right]^{1/k} = \lim_{k\to\infty} \frac{|x-2|}{4} = \frac{|x-2|}{4}.$$

In this case, ρ depends on the value of x. For absolute convergence, x must satisfy

$$\rho = \frac{|x-2|}{4} < 1,$$

which implies that $|x-2| < 4$. Using standard technique for solving inequalities, the solution set is $-4 < x-2 < 4$, or $-2 < x < 6$. We conclude that the series converges on $(-2, 6)$ by Theorem 1.6. The Root Test does not give information about convergence at the endponits $x = -2$ and $x = 6$, because at these points, the Root Test results in $\rho = 1$. To test for convergence at the endpoints, we substitute each endpoint into the series and carry out separate tests. At $x = -2$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{4^k} = \sum_{k=0}^{\infty} \frac{4^k}{4^k} = \sum_{k=0}^{\infty} 1 = \lim_{n\to\infty} \sum_{k=0}^{n} 1 = \lim_{n\to\infty} n+1 = \infty.$$

Therefore, the series diverges at the left endpoint (or by Divergence Test, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} 1 = 1 \neq 0$). At $x = 6$, the power series is

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{4^k} = \sum_{k=0}^{\infty} (-1)^k \frac{4^k}{4^k} = \sum_{k=0}^{\infty} (-1)^k.$$

By Divergence Test, the series diverges at the right endpoint. Hence, the interval of convergence is $(-2, 6)$, excluding the endpoints, and the radius of convergence is $R = 4$.

(c) In this case, the Ratio Test is preferable:

$$r = \lim_{k\to\infty} \frac{|(k+1)!\, x^{k+1}|}{|k!\, x^k|} = \lim_{k\to\infty} (k+1)|x| = |x| \lim_{k\to\infty} (k+1) = \begin{cases} \infty & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We see that $r > 1$ for all $x \neq 0$, so the series diverges on $(-\infty, 0)$ and $(0, \infty)$. The only way to satisfy $r < 1$ is to take $x = 0$, in which case the power series has a value of 0. The interval of convergence of the power series consists of the single point $x = 0$, and the radius of convergence is $R = 0$.

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