# MATH 2205 - Calculus II Lecture Notes 17

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## 1 Integration Techniques

### 1.1 Basic Approaches

**Proposition 1.1** (Basic Integration Formulas)**.**

(a) $\int k\,dx = kx + C$, $k \in \mathbb{R}$ ($k$ is real).

(b) $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1 \in \mathbb{R}$.

(c) $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$.

(d) $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$.

(e) $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$.

(f) $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$.

(g) $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$.

(h) $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$.

(i) $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$.

(j) $\int \frac{1}{x}\,dx = \ln|x| + C$.

(k) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$.

(l) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$.

(m) $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$, $a > 0$.

### 1.2 Integration by Parts

**Theorem 1.1** (Integration by Parts)**.** Suppose that $u$ and $v$ are differentiable functions. Then

$$\int u\,dv = uv - \int v du.$$

**Theorem 1.2** (Integration by Parts for Definite Integrals)**.** Let $u$ and $v$ be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

## 1.3 Trigonometric Integrals

### 1.3.1 Integrating Powers of $\sin x$ or $\cos x$

**Procedure 1.1.** Strategies for evaluating integrals of the form $\int \sin^m x\,dx$ or $\int \cos^n x\,dx$, where $m$ and $n$ are positive integers, using trigonometric identities.

(a) Integrals involving odd powers of $\cos x$ (or $\sin x$) are most easily evaluated by splitting off a single factor of $\cos x$ (or $\sin x$). For example, rewrite $\cos^5 x$ as $\cos^4 x \cdot \cos x$.

(b) With even positive powers of $\sin x$ or $\cos x$, we use the half-angle formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } \cos^2\theta = \frac{1+\cos 2\theta}{2},$$

to reduce the powers in the integrand.

### 1.3.2 Integrating Products of Powers of $\sin x$ and $\cos x$

**Procedure 1.2.** Strategies for evaluating integrals of the form $\int \sin^m x \cos^n x\,dx$.

(a) When $m$ is odd and positive, $n$ real. Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.

(b) When $n$ is odd and positive, $m$ real. Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

(c) When $m$, $n$ are both even and nonnegative. Use half-angle formulas to transform the integrand into polynomial in $\cos 2x$ and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

**Proposition 1.2** (Reduction Formulas)**.** Assume $n$ is a positive integer.

(a) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

(b) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

(c) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, n \neq 1.$

(d) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, n \neq 1.$

## 1.4 Trigonometric Substitutions

**Proposition 1.3.** The integral contains $a^2 - x^2$. Let $x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$ for $|x| \le a$. Then $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\cos^2\theta) = a^2\cos^2\theta$.

## 1.5 Partial Fractions

**Procedure 1.3** (Partial Fractions with Simple Linear Factors)**.** Suppose $f(x) = p(x)/q(x)$, where $p$ and $q$ are polynomials with no common factors and with the degree of $p$ less than the degree of $q$. Assume that $q$ is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

(a) *Factor the denominator* $q$ in the form $(x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, \ldots, r_n$ are real numbers.
(b) *Partial fraction decomposition.* Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

(c) *Clear denominators.* Multiply both sides of the equation in Step (b) by $q(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$, which produces conditions for $A_1, \ldots, A_n$.
(d) *Solve for coefficients.* Equate like powers of $x$ in Step (c) to solve for the undetermined coefficients $A_1, \ldots, A_n$.

**Procedure 1.4** (Partial Fractions for Repeated Linear Factors)**.** Suppose the repeated linear factor $(x - r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x - r)$ up to and including the $m$th power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m},$$

where $A_1, \ldots, A_m$ are constants to be determined.

**Procedure 1.5** (Partial Fractions with Simple Irreducible Quadratic Factors)**.** Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where $A$ and $B$ are unknown coefficients to be determined.

**Proposition 1.4.** The quadratic polynomial $ax^2 + bx + c$ is irreducible if and only if its discriminant is negative, i.e.,

$$\Delta = b^2 - 4ac < 0.$$

**Proposition 1.5** (Partial Fraction Decomposition)**.** Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator $q$ has been factored completely over the real numbers and $m$ is a positive integer.

(a) *Simple linear factor.* A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x - r}$.
(b) *Repeated linear factor.* A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m}.$$

(c) *Simple irreducible quadratic factor.* An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c}.$$

(d) *Repeated irreducible quadratic factor.* An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}.$$

## 1.6 Numerical Integration

**Definition 1.1** (Absolute and Relative Error)**.** Suppose $c$ is a computed numerical solution to a problem having an exact solution $x$. There are two common meaasures of the error in $c$ as an approximation to $x$:

$$\textbf{absolute error} = |c - x|$$

and

$$\textbf{relative error} = \frac{c-x}{x}, (\text{if } x \neq 0).$$

**Definition 1.2.** Suppose $f$ is defined an integrable on $[a, b]$. The *Midpoint Rule approximation* to $\int_a^b f(x)\,dx$ using $n$ equally spaced subintervals on $[a, b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x = \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x,$$

where $\Delta x = (b-a)/n$, $x_0 = a$, $x_k = a + k\Delta x$, and $m_k = (x_{k-1} + x_k)/2 = a + (k - 1/2)\Delta x$ is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, \ldots, n$.

**Definition 1.3** (Trapezoid Rule)**.** Suppose $f$ is defined and integrable on $[a, b]$. The *Trapezoid Rule approximation* to $\int_a^b f(x)\,dx$ using $n$ equally spaced subintervals on $[a, b]$ is

$$T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x.$$

where $\Delta x = (b-a)/n$ and $x_k = a + k\Delta x$, for $k = 0, 1, 2, \ldots, n$.

**Definition 1.4** (Simpson's Rule)**.** Suppose $f$ is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The *Simpson's Rule approximation* to $\int_a^b f(x)\,dx$ using $n$ equally spaced subintervals on $[a, b]$ is

$$\begin{aligned} S(n) &= [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]\frac{\Delta x}{3} \\ &= \sum_{k=0}^{n/2-1} [f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})]\frac{\Delta x}{3}. \end{aligned}$$

where $n$ is an even integer, $\Delta x = (b-a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

## 1.7 Improper Integrals

**Definition 1.5** (Improper Integrals over Infinite Intervals)**.**

(a) If $f$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

(b) If $f$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx.$$

(c) If $f$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{c} f(x)\,dx + \lim_{b\to\infty}\int_{c}^{b} f(x)\,dx,$$

where $c$ is any real number.

If the limits in the above cases exist, then the improper integrals *converge*; otherwise, they *diverge*.

**Definition 1.6** (Improper Integrals with an Unbounded Integrand)**.**

(a) Suppose $f$ is continous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_{a}^{b} f(x)\,dx = \lim_{c\to a^+}\int_{c}^{b} f(x)\,dx.$$

(b) Suppose $f$ is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then

$$\int_{a}^{b} f(x)\,dx = \lim_{c\to b^-} f(x)\,dx.$$

(c) Suppose $f$ is continuous on $[a, b]$ except at the interior point $p$ where $f$ is unbounded. Then

$$\int_{a}^{b} f(x)\,dx = \lim_{c\to p^-}\int_{a}^{c} f(x)\,dx + \lim_{d\to p^+}\int_{d}^{b} f(x)\,dx.$$

If the limits in above cases exist, then the improper integrals *converge*; otherwise, they *diverge*.

## 1.8 Introduction to Differential Equations

**Definition 1.7.**

(a) The *order* of a differential equation is the highest order appearing on a derivative in the equation. For example, the equations $y' + 4y = \cos x$ and $y' = 0.1y(100 - y)$ are first order, and $y'' + 16y = 0$ is second order.

(b) *Linear* differential equations (first- and second-order) have the form

$$y'(x) + p(x)y(x) = f(x) \text{ and } y''(x) + p(x)y'(x) + q(x)y(x) = f(x),$$

where $p$, $q$, and $f$ are given functions that depend only on the independent variable $x$.

(c) A differential equation is often accompanied by *initial conditions* that specify the values of $y$, and possibly its derivatives, at a particular point. In general, an $n$th-order equation reqruires $n$ initial conditions.

(d) A differntial equation, together with the appropriate number of initial conditions, is called an *initial value problem*. A typical first-order initial value problem has the form

$$\begin{aligned} y'(t) &= F(t, y) && \text{Differential equation} \\ y(0) &= A && \text{Initial condition} \end{aligned}$$

where $A$ is given and $F$ is a given expression that involves $t$ and/or $y$,

**Proposition 1.6** (Solution of a First-Order Linear Differential Equation)**.** The general solution of the first-order equation $y'(t) = ky + b$, where $k$ and $b$ are specified real numbers, is $y = Ce^{kt} - b/k$, where $C$ is an arbitrary constant. Given an initial condition, the value of $C$ may be determined.

**Definition 1.8** (Separable First-Order Differential Equations)**.** If the first-order differential equation can be written in the form $g(y)y'(t) = h(t)$, in which the terms that involve $y$ appear on one side of the equation *separated* from the terms that involve $t$, is said to be *separable*. We can solve the equaiton by integrating both sides of the equation with respect to $t$:

$$\int g(y)y'(t)\,dt = \int h(t)\,dt \implies \int g(y)\,dy = \int h(t)\,dt.$$

# 2 Sequences and Infinite Series

## 2.1 Sequences

**Definition 2.1** (Sequence)**.** A *sequence* $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a *recurrence relation* of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where $a_1$ is given. A sequence may also be defined with an *explicit formula* of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

**Definition 2.2** (Limit of a Sequence)**.** If the terms of a sequence $\{a_n\}$ approach a unique number $L$ as $n$ increases – that is, if $a_n$ can be made arbitrarily close to $L$ by taking $n$ sufficiently large – then we say $\lim\limits_{n\to\infty} a_n = L$ exists, and the sequence *converges* to $L$. If the terms of the sequence do not approach a single number as $n$ increases, the sequence has no limit, and the sequence *diverges*.

**Theorem 2.1** (Limits of Sequences from Limits of Functions)**.** Suppose $f$ is a function such that $f(n) = a_n$ for all positive integers $n$. If $\lim\limits_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also $L$.

**Theorem 2.2** (Limit Laws for Sequences)**.** Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits $A$ and $B$, respectively. Then

(a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$.
(b) $\lim\limits_{n\to\infty} ca_n = cA$, where $c$ is a real number.
(c) $\lim\limits_{n\to\infty} a_n b_n = AB$.
(d) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ provided $B \neq 0$.

**Definition 2.3** (Terminology for Sequences)**.**

(a) $\{a_n\}$ is *increasing* if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$.
(b) $\{a_n\}$ is *nondecreasing* if $a_{n+1} \geq a_n$; for example, $\{0, 1, 1, 1, 2, 2, 3, \ldots\}$.
(c) $\{a_n\}$ is *decreasing* if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -2, \ldots\}$.
(d) $\{a_n\}$ is *nonincreasing* if $a_{n+1} \leq a_n$; for example, $\{2, 1, 1, 0, -2, -2, -3, \ldots\}$.
(e) $\{a_n\}$ is *monotonic* if it is either nonincreasing or nondecreasing (it moves in one direction).
(f) $\{a_n\}$ is *bounded* if there is number $M$ such that $|a_n| \leq M$, for all relevant values of $n$.

**Theorem 2.3** (Squeeze Theorem for Sequences)**.** Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \le b_n \le c_n$ for all integers $n$ greater than some index $N$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

**Theorem 2.4** (Bounded Monotonic Sequences)**.** A bounded monotonic sequence converges.

**Theorem 2.5** (Growth Rates of Sequences)**.** The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers $p, q, r, s$ and $b > 1$.

## 2.2 Limits of Functions

**Theorem 2.6** (Limits of Linear Functions)**.** Let $a$, $b$, and $m$ be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x\to a} f(x) = f(a) = ma + b.$$

**Theorem 2.7** (Limit Laws)**.** Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where $c$ is a real number, and $m > 0$ and $n > 0$ are integers.

(a) *Sum*: $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
(b) *Difference*: $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
(c) *Constant multiple*: $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$.
(d) *Product*: $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$.
(e) *Quotient*: $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \ne 0$.
(f) *Power*: $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$.

**Theorem 2.8** (Limits of Polynomial and Rational Functions)**.** Assume $p$ and $q$ are polynomials and $a$ is a constant.

(a) Polynomial functions: $\lim_{x\to a} p(x) = p(a)$.
(b) Rational functions: $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \ne 0$.

**Theorem 2.9** (The Squeeze Theorem)**.** Assume the function $f$, $g$, and $h$ satisfy $f(x) \le g(x) \le h(x)$ for all values of $x$ near $a$, except possibly at $a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

**Theorem 2.10** (Limits at Infinity of Powers and Polynomials)**.** Let $n$ be a positive integer and let $p$ be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \ne 0$.

(a) $\lim_{x\to\pm\infty} x^n = \infty$ when $n$ is even.
(b) $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when $n$ is odd.

(c) $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0.$

(d) $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient $a_n$.

**Theorem 2.11** (End Behavior of $e^x$, $e^{-x}$, and $\ln x$)**.** The end behavior for $e^x$ and $e^{-x}$ on $(-\infty,\infty)$ and $\ln x$ on $(0,\infty)$ is given by the following limits (see Figure 1):

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0$$
$$\lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$
$$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty.$$

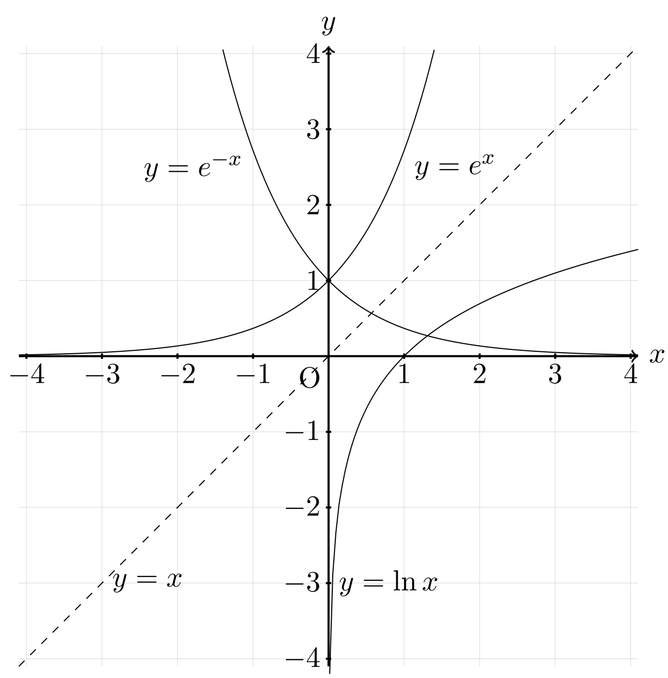


Figure 1: Graphs of $e^x$, $e^{-x}$, $\ln x$: $y = e^{-x}$ and $y = e^x$ are symmetric about $y$-axis, and $y = e^x$ and $y = \ln x$ are symmetric about $y = x$.

**Theorem 2.12** (L'Hôpital's Rule)**.** Suppose $f$ and $g$ are differentiable on an open interval $I$ containing $a$ with $g'(x) \neq 0$ on $I$ when $x \neq a$.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is repaced with $x \to \pm\infty$, $x \to a^+$, $x \to a^-$.

(b) If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is repaced with $x \to \pm\infty$, $x \to a^+$, $x \to a^-$.

## 2.3 Infinite Series

**Definition 2.4** (Infinite series)**.** Given a sequence $\{a_1, a_2, a_3, \dots, \}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an *infinite series*. The *sequence of partial sums* $\{S_n\}$ associated with this series has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \text{for } n = 1, 2, 3, \dots \end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit $L$, the infinite series *converges* to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also *diverges.*

## 2.4 Geometric Sequences and Geometric Series

**Definition 2.5** (Geometric Sequences)**.** A sequence has the form $\{r^n\}$ or $\{ar^n\}$, where the ratio $r$, $a$ are real numbers, is called a geometric sequence.

**Theorem 2.13** (Geometric Sequences)**.** Let $r$ be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1, \\ 1 & \text{if } r = 1, \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.

**Theorem 2.14** (Geometric Series)**.** Let $a \neq 0$ and $r$ be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges. More generally,

$$\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}.$$

# 3 Algebra

## 3.1 Exponents and Radicals

(a) $\frac{1}{x^a} = x^{-a}$.
(b) $\sqrt[n]{x} = x^{1/n}$.
(c) $x^{a+b} = x^a x^b$.
(d) $x^{a-b} = \frac{x^a}{x^b}$.
(e) $x^{ab} = (x^a)^b$.
(f) $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.
(g) $(xy)^a = x^a y^a$.
(h) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$.

## 3.2 Logarithm

(a) $y = a^x \implies x = \log_a y$.
(b) $\log_e x = \ln x$.
(c) $\log_b(xy) = \log_b x + \log_b y$.
(d) $\log_b \frac{x}{y} = \log_b x - \log_b y$.
(e) $\log_b(x^p) = p \log_b x$.
(f) $\log_b(x^{1/p}) = \frac{1}{p} \log_b x$.
(g) $\log_b x = \frac{\log_k x}{\log_k b}$.

## 3.3 Factoring Formulas

(a) $a^2 - b^2 = (a-b)(a+b)$.
(b) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
(c) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$.

## 3.4 Binomials

(a) $(a \pm b)^2 = a^2 \pm 2ab + b^2$.
(b) $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.

## 3.5 Completing the Square

(a) $(x \pm p)^2 = x^2 \pm 2px + p^2$.
(b)

$$\begin{aligned} x^2 \pm bx + c &= x^2 \pm 2\frac{b}{2}x + c \\ &= x^2 \pm 2\frac{b}{2}x + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 \\ &= \left(x \pm \frac{b}{2}\right)^2 + c - \frac{b^2}{4}. \end{aligned}$$

(c)

$$\begin{aligned}
ax^2 \pm bx + c &= a\left(x^2 \pm \frac{b}{a}x\right) + c \\
&= a\left(x^2 \pm 2\frac{b}{2a}x\right) + c \\
&= a\left[x^2 \pm 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right] + c - a\left(\frac{b}{2a}\right)^2 \\
&= a\left(x \pm \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= (\sqrt{a})^2\left(x \pm \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= \left(\sqrt{a}x \pm \frac{\sqrt{a}b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= \left(\sqrt{a}x \pm \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a}.
\end{aligned}$$

## 3.6 Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$