MATH 2205 - Calculus II Lecture Notes 15

Last update: June 20, 2019

1 Differential Equations

1.1 Review of Linear Differential Equations

Definition 1.1.

(a) The *order* of a differential equation is the highest order appearing on a derivative in the equation. For example, the equations $y' + 4y = \cos x$ and $y' = 0.1y(100 - y)$ are first order, and $y'' + 16y = 0$ is second order.

(b) *Linear* differential equations (first- and second-order) have the form

$$y'(x) + p(x)y(x) = f(x) \text{ and } y''(x) + p(x)y'(x) + q(x)y(x) = f(x),$$

where p, q, and f are given functions that depend only on the independent variable x.

(c) A differential equation is often accompanied by *initial conditions* that specify the values of y, and possibly its derivatives, at a particular point. In general, an nth-order equation reqruires n initial conditions.

(d) A differntial equation, together with the appropriate number of initial conditions, is called an *initial value problem.* A typical first-order initial value problem has the form

$$\begin{aligned} y'(t) &= F(t, y) && \text{Differential equation} \\ y(0) &= A && \text{Initial condition} \end{aligned}$$

where A is given and F is a given expression that involves t and/or y,

Proposition 1.1 (Solution of a First-Order Linear Differential Equation)**.** The general solution of the first-order equation $y'(t) = ky + b$, where k and b are specified real numbers, is $y = Ce^{kt} - b/k$, where C is an arbitrary constant. Given an initial condition, the value of C may be determined.

Definition 1.2 (Separable First-Order Differential Equations)**.** If the first-order differential equation can be written in the form $g(y)y'(t) = h(t)$, in which the terms that involve y appear on one side of the equation *separated* from the terms that involve t, is said to be *separable.* We can solve the equaiton by integrating both sides of the equation with respect to t:

$$\int g(y)y'(t)\,dt = \int h(t)\,dt \implies \int g(y)\,dy = \int h(t)\,dt.$$

Example 1.1 (A separable equation)**.** Find the funciton that satisfies the inital value problem

$$\frac{dy}{dx} = y^2 e^{-x}, \quad y(0) = \frac{1}{2}.$$

Solution. The equation can be written in separable form by dividing form by dividing both sides of the equation by y^2 to give $\frac{y'}{y^2} = e^{-x}$. We can integrate both sides of the equation with respect to x and evaluate the resulting integrals.

$$\int \frac{1}{y^2} \underbrace{y'(x)\,dx}_{dy} = \int e^{-x}\,dx$$

$$\int y^{-2}\,dy = -\int e^{-x} \; -dx$$

$$-y^{-1} = -e^{-x} + C$$

$$y = \frac{1}{e^{-x} - C}.$$

The initial condition $y(0) = 1/2$ implies that

$$y(0) = \frac{1}{e^0 - C} = \frac{1}{1 - C} = \frac{1}{2} \implies C = -1.$$

So the solution of the initial value problem is $y = \dfrac{1}{e^{-x} + 1}$. □

2 Sequences and Infinite Series

2.1 An Overview

Definition 2.1 (Sequence)**.** A *sequence* $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a *recurrence relation* of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an *explicit formula* of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Example 2.1 (Explicit formulas)**.** Use the explicit formula for $\{a_n\}_{n=1}^{\infty}$ to write the first four terms of each sequence. Sketch a graph of the sequence.

(a) $a_n = \dfrac{1}{2^n}$.

(b) $a_n = \dfrac{(-1)^n n}{n^2 + 1}$.

Solution.

(a)

$$a_1 = \left.\frac{1}{2^n}\right|_{n=1} = \frac{1}{2^1} = \frac{1}{2},$$

$$a_2 = \left.\frac{1}{2^n}\right|_{n=2} = \frac{1}{2^2} = \frac{1}{4},$$

$$a_3 = \left.\frac{1}{2^n}\right|_{n=3} = \frac{1}{2^3} = \frac{1}{8},$$

$$a_4 = \left.\frac{1}{2^n}\right|_{n=4} = \frac{1}{2^4} = \frac{1}{16}.$$

(b)

$$
\begin{aligned}
a_1 &= \left.\frac{(-1)^n n}{n^2+1}\right|_{n=1} = \frac{(-1)^1 \times 1}{1^2+1} = \frac{-1}{2}, \\
a_2 &= \left.\frac{(-1)^n n}{n^2+1}\right|_{n=2} = \frac{(-1)^2 \times 2}{2^2+1} = \frac{2}{5}, \\
a_3 &= \left.\frac{(-1)^n n}{n^2+1}\right|_{n=3} = \frac{(-1)^3 \times 3}{3^2+1} = \frac{-3}{10}, \\
a_4 &= \left.\frac{(-1)^n n}{n^2+1}\right|_{n=4} = \frac{(-1)^4 \times 4}{4^2+1} = \frac{4}{17}.
\end{aligned}
$$

□

Example 2.2 (Recurrence relations)**.** Use the recurrence relation for $\{a_n\}_{n=1}^{\infty}$ to write the first four terms of the sequences

$$a_{n+1} = 2a_n + 1, a_1 = 1 \text{ and } a_{n+1} = 2a_n + 1, a_1 = -1.$$

SOLUTION.

(a)

$$
\begin{aligned}
a_1 &= 1, \\
a_2 &= 2a_n + 1\Big|_{n=1} = 2a_1 + 1 = 2 \times 1 + 1 = 3, \\
a_3 &= 2a_n + 1\Big|_{n=2} = 2a_2 + 1 = 2 \times 3 + 1 = 7, \\
a_4 &= 2a_n + 1\Big|_{n=3} = 2a_3 + 1 = 2 \times 7 + 1 = 15.
\end{aligned}
$$

(b)

$$
\begin{aligned}
a_1 &= -1, \\
a_2 &= 2a_n + 1\Big|_{n=1} = 2a_1 + 1 = 2 \times (-1) + 1 = -1, \\
a_3 &= 2a_n + 1\Big|_{n=2} = 2a_2 + 1 = 2 \times (-1) + 1 = -1, \\
a_4 &= 2a_n + 1\Big|_{n=3} = 2a_3 + 1 = 2 \times (-1) + 1 = -1.
\end{aligned}
$$

□

Example 2.3 (Working with sequences)**.** Consider the following sequences: $\{a_n\} = \{-2, 5, 12, 19, \ldots\}$ and $\{b_n\} = \{3, 6, 12, 24, 48, \ldots\}$.

(a) Find the next two terms of the sequnces.
(b) Find a recurrence relation that generates the sequence.
(c) Find an explicit formula for the nth term of the sequence.

SOLUTION.

(a)
- Observe that the difference between a_{n+1} and a_n for $n = 1, 2, 3$ is 7, i.e.,

$$a_{n+1} - a_n = 7 \implies a_{n+1} = a_n + 7, \text{ for } n = 1, 2, 3.$$

Then $a_5 = a_4 + 7 = 19 + 7 = 26$, and $a_6 = a_5 + 7 = 33$.
- Observe that the ratio between b_{n+1} and b_n for $n = 1, 2, 3$ is 2, i.e.,

$$\frac{b_{n+1}}{b_n} = 2 \implies b_{n+1} = 2b_n \text{ for } n = 1, 2, 3.$$

Then $b_5 = 2b_4 = 2 \times 48 = 96$, and $b_6 = 2b_5 = 2 \times 96 = 192$.

(b) By the above observation, we have

$$a_{n+1} = a_n + 7, a_1 = -2 \text{ and } b_{n+1} = 2b_n, b_1 = 3.$$

(c) Apply the recurrence relation $a_{n+1} = a_n + 7, a_1 = -2$ to a_n $n - 1$ times, we can get the explicit formulas for a_n as follows

$$\begin{aligned}
a_n &= a_{n-1} + 7 \\
&= (a_{n-2} + 7) + 7 && [a_{n-1} = a_{n-2} + 7] \\
&= a_{n-2} + 2 \times 7 \\
&= (a_{n-3} + 7) + 2 \times 7 && [a_{n-2} = a_{n-3} + 7] \\
&= a_{n-3} + 3 \times 7 \\
&\vdots \\
&= a_{n-(n-1)} + (n - 1) \times 7 \\
&= a_1 + 7(n - 1) \\
&= -2 + 7(n - 1) \\
&= -9 + 7n,
\end{aligned}$$

Similarly, we can apply recurrence relation $b_{n+1} = 2b_n$ to b_n $n - 1$ times to get the corresponding formula for b_n as below,

$$\begin{aligned}
b_n &= 2b_{n-1} \\
&= 2(2b_{n-2}) && [b_{n-1} = 2b_{n-2}] \\
&= 2^2 b_{n-2} \\
&= 2^2(2b_{n-3}) && [b_{n-2} = 2b_{n-3}] \\
&= 2^3 b_{n-3} \\
&\vdots \\
&= 2^{n-1} b_{n-(n-1)} \\
&= 2^{n-1} b_1 \\
&= 3 \times 2^{n-1}.
\end{aligned}$$

□

Definition 2.2 (Limit of a Sequence)**.** If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases – that is, if a_n can be made arbitrarily close to L by taking n sufficiently large – then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence *converges* to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence *diverges*.

Example 2.4 (Limits of sequences)**.** Write the first four terms of each sequence. If you believe the sequence converges, make a conjecture about its limit. If the sequence appears to diverge, explain why.

(a) $\left\{\dfrac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$.
(b) $\{\cos n\pi\}_{n=1}^{\infty}$.
(c) $\{a_n\}_{n=1}^{\infty}$, where $a_{n+1} = -2a_n, a_1 = 1$.

Solution.

(a) The first four terms of the sequence are

$$\left\{\frac{(-1)^1}{1^2+1}, \frac{(-1)^2}{2^2+1}, \frac{(-1)^3}{3^2+1}, \frac{(-1)^4}{4^2+1}, \ldots\right\} = \left\{-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, \ldots\right\}$$

Observe that $-\dfrac{1}{n^2} < \dfrac{(-1)^n}{n^2+1} < \dfrac{1}{n^2}$, and

$$\lim_{n\to\infty} -\frac{1}{n^2} = 0 \text{ and } \lim_{n\to\infty} \frac{1}{n^2} = 0 \implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0.$$

We can also graph the sequence (see Figure 1).

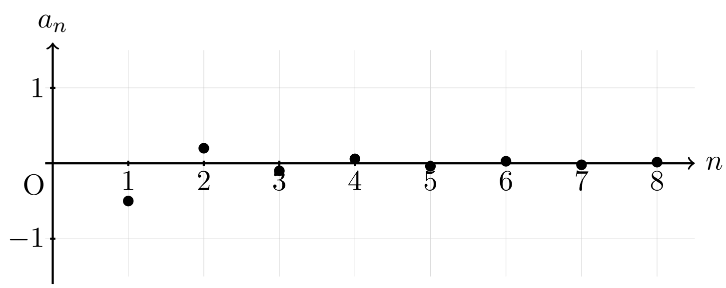

Figure 1: $\left\{\dfrac{(-1)^n}{n^2+1}\right\}_{n=1}^{8}$

(b) The first four terms of the sequence are

$$\{\cos 1\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \ldots\} = \{-1, 1, -1, 1, \ldots\}.$$

In this case, the terms of the sequence alternate between -1 and $+1$, and never approach a single value. Therefore, the sequence diverges. We can also graph the sequence (see Figure 2).

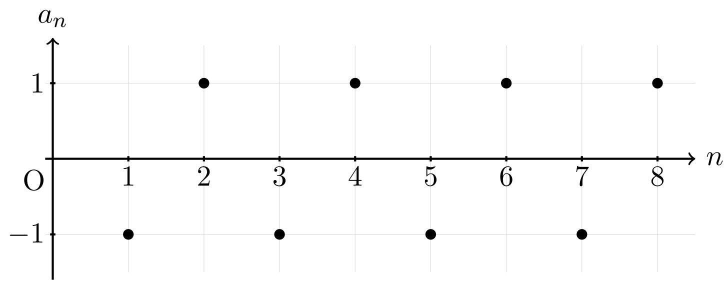

Figure 2: $\{\cos n\pi\}_{n=1}^{8}$

(c) First let us derive the explicit formula for the sequence using the recurrence relation $a_{n+1} = -2a_n, a_1 = 1$, apply the relation to a_n $(n-1)$ times we have

$$a_n = -2a_{n-1} = -2(-2a_{n-2}) = (-2)^2 a_{n-2} = (-2)^3 a_{n-3} = \cdots = (-2)^{n-1} a_1 = (-2)^{n-1}.$$

Then the first four terms of the sequence are

$$\begin{aligned}\{a_1, a_2, a_3, a_4, \ldots\} &= \{(-2)^{1-1}, (-2)^{2-1}, (-2)^{3-1}, (-2)^{4-1}, \ldots\} \\ &= \{(-2)^0, (-2)^1, (-2)^2, (-2)^3, \ldots\} \\ &= \{1, -2, 4, -8, \ldots\}.\end{aligned}$$

The magnitude of the terms increase without bound, the sequence thus diverges.

$\square$