MATH 2205 - Calculus II Lecture Notes 14

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1 Integration Techniques

1.1 Review of Improper Integrals

Definition 1.1 (Improper Integrals over Infinite Intervals)**.**

(a) If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

(b) If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

(c) If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx,$$

where c is any real number.

If the limits in the above cases exist, then the improper integrals *converge*; otherwise, they *diverge*.

Definition 1.2 (Improper Integrals with an Unbounded Integrand)**.**

(a) Suppose f is continous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

(b) Suppose f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} f(x)\,dx.$$

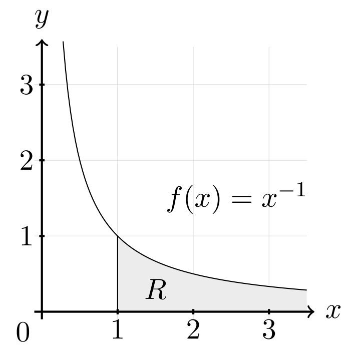
(c) Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \lim_{c\to p^-} \int_a^c f(x)\,dx + \lim_{d\to p^+} \int_d^b f(x)\,dx.$$

If the limits in above cases exist, then the improper integrals *converge*; otherwise, they *diverge*.

Example 1.1 (Solids of revolution)**.** Let R be the region bounded by the graph of $y = x^{-1}$ and the x-axis, for $x \geq 1$.

(a) What is the volume of the solid generated when R is revolved about the x-axis?



(b) What is the volume of the solid generated when R is revolved about the y-axis?

Solution.

(a) Disk method.

$$
\begin{aligned}
V &= \int_a^b \pi f(x)^2\,dx \\
&= \int_1^\infty \pi (x^{-1})^2\,dx \\
&= \pi \int_1^\infty x^{-2}\,dx \\
&= \pi \lim_{b\to\infty} \int_1^b x^{-2}\,dx \\
&= \pi \lim_{b\to\infty} (-x^{-1}) \Big|_1^b \\
&= -\pi \lim_{b\to\infty} x^{-1} \Big|_1^b \\
&= -\pi \lim_{b\to\infty} (b^{-1} - 1) \\
&= -\pi(-1) \\
&= \pi.
\end{aligned}
$$

The volume of the solid when R is revolved about the x-axis is π.

(b) Shell method.

$$\begin{aligned} V &= \int_a^b 2\pi x f(x)\,dx \\ &= \int_1^\infty 2\pi x x^{-1}\,dx \\ &= 2\pi \int_1^\infty 1\,dx \\ &= 2\pi \lim_{b\to\infty} \int_1^b 1\,dx \\ &= 2\pi \lim_{b\to\infty} x\Big|_1^b \\ &= 2\pi \lim_{b\to\infty} (b-1) \\ &= \infty. \end{aligned}$$

In this case, the volume of the solid when R is revolved about the y-axis diverges.

$\square$

1.2 Review for Computing Limits

Theorem 1.1 (Limits of Linear Functions)**.** Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x\to a} f(x) = f(a) = ma + b.$$

Theorem 1.2 (Limit Laws)**.** Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where c is a real number, and $m > 0$ and $n > 0$ are integers.

(a) *Sum*: $\lim\limits_{x\to a}[f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$.

(b) *Difference*: $\lim\limits_{x\to a}[f(x) - g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$.

(c) *Constant multiple*: $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$.

(d) *Product*: $\lim\limits_{x\to a}[f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right]$.

(e) *Quotient*: $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.

(f) *Power*: $\lim\limits_{x\to a}[f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$.

(g) *Fractional power*: $\lim\limits_{x\to a}[f(x)]^{n/m} = \left[\lim\limits_{x\to a} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms.

Theorem 1.3 (Limits of Polynomial and Rational Functions)**.** Assume p and q are polynomials and a is a constant.

(a) Polynomial functions: $\lim_{x\to a} p(x) = p(a)$.

(b) Rational functoins: $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$, provided $q(a) \neq 0$.

Theorem 1.4 (The Squeeze Theorem)**.** Assume the function f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Theorem 1.5 (Limits at Infinity of Powers and Polynomials)**.** Let n be a positive integer and let p be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

(a) $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
(b) $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.
(c) $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$.
(d) $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Example 1.2. Let $p(x) = \displaystyle\sum_{k=0}^{n} a_k x^k = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = \displaystyle\sum_{k=0}^{m} b_k x^k = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$. What is $\displaystyle\lim_{x\to\infty} \frac{p(x)}{q(x)}$?

Solution. It is known that $\lim_{x\to\infty} x^n = \begin{cases} \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$.

$$
\begin{aligned}
\lim_{x\to\infty} \frac{p(x)}{q(x)} &= \lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} \\
&= \lim_{x\to\infty} \frac{a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n}}{b_m x^{m-n} + b_{m-1}x^{m-n-1} + \cdots + b_1 x^{1-n} + b_0 x^{-n}} && [\text{divide both top and bottom by } x^n] \\
&= \frac{\lim_{x\to\infty} a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n}}{\lim_{x\to\infty} b_m x^{m-n} + b_{m-1}x^{m-n-1} + \cdots + b_1 x^{1-n} + b_0 x^{-n}} && [\text{Quotient rule}] \\
&= \frac{a_n}{\lim_{x\to\infty} b_m x^{m-n} + b_{m-1}x^{m-n-1} + \cdots + b_1 x^{1-n} + b_0 x^{-n}} \\
&= \begin{cases} a_n/b_m & \text{if } m = n, \\ \infty & \text{if } m < n, \\ 0 & \text{if } m > n. \end{cases}
\end{aligned}
$$

□

Theorem 1.6 (End Behavior of e^x, e^{-x}, and $\ln x$)**.** The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits (see Figure 1):

$$
\begin{aligned}
&\lim_{x\to\infty} e^x = \infty &\qquad &\lim_{x\to-\infty} e^x = 0 \\
&\lim_{x\to\infty} e^{-x} = 0 &\qquad &\lim_{x\to-\infty} e^{-x} = \infty \\
&\lim_{x\to 0^+} \ln x = -\infty &\qquad &\lim_{x\to\infty} \ln x = \infty.
\end{aligned}
$$

Theorem 1.7 (L'Hôpital's Rule)**.** Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$.

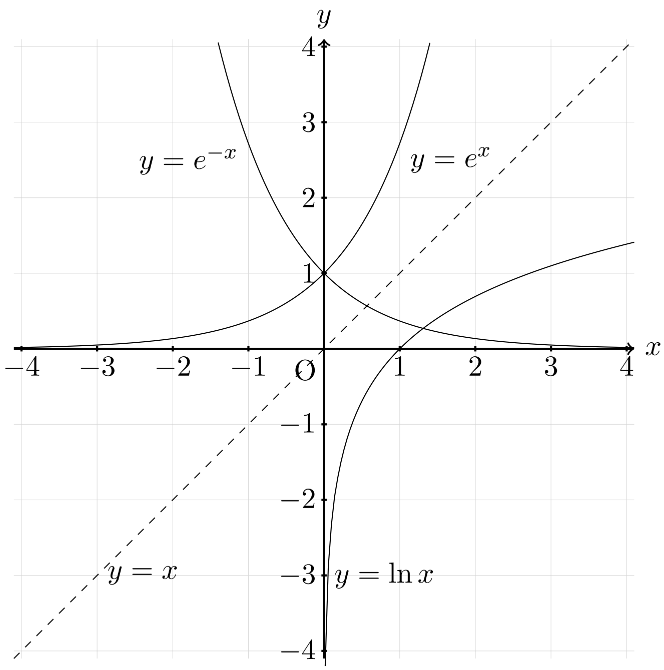


Figure 1: Graphs of e^x, e^{-x}, $\ln x$: $y = e^{-x}$ and $y = e^x$ are symmetric about y-axis, and $y = e^x$ and $y = \ln x$ are symmetric about $y = x$.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is repaced with $x \to \pm\infty$, $x \to a^+$, $x \to a^-$.

(b) If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is repaced with $x \to \pm\infty$, $x \to a^+$, $x \to a^-$.

1.3 Introduction to Differential Equations

Example 1.3 (Differential Equations)**.** Examples of differential equations:

(a) $y''(x) + 16y = 0$.
(b) $\dfrac{dy}{dx} + 4y = \cos x$.
(c) $y'(t) = 0.1y(100 - y)$.

In each case, the goal is to find solution of the equation – that is, functions y that satisfy the equation.

Definition 1.3.

(a) The *order* of a differential equation is the highest order appearing on a derivative in the equation. For example, the equations $y' + 4y = \cos x$ and $y' = 0.1y(100 - y)$ are first order, and $y'' + 16y = 0$ is second order.

(b) *Linear* differential equations (first- and second-order) have the form

$$y'(x) + p(x)y(x) = f(x) \text{ and } y''(x) + p(x)y'(x) + q(x)y(x) = f(x),$$

where p, q, and f are given functions that depend only on the independent variable x.

(c) A differential equation is often accompanied by *initial conditions* that specify the values of y, and possibly its derivatives, at a particular point. In general, an nth-order equation reqruires n initial conditions.

(d) A differntial equation, together with the appropriate number of initial conditions, is called an *initial value problem.* A typical first-order initial value problem has the form

$$\begin{aligned} y'(t) &= F(t, y) && \text{Differential equation} \\ y(0) &= A && \text{Initial condition} \end{aligned}$$

where A is given and F is a given expression that involves t and/or y,

Example 1.4 (An initial value problem)**.** Solve the initial value problem

$$y'(t) = 10e^{-t/2}, y(0) = 4.$$

Solution. Notice that the right side of the equation depends only on t. The solution is found by integrating both sides of the differential equation with respect to t:

$$\int y'(t)\,dt = \int 10e^{-t/2}dt \implies y = -20e^{-t/2} + C.$$

We have found the general solution, which involves one arbitrary constant. To determine its value, we use the initial condition by substituting $t = 0$ and $y = 4$ into the general solution:

$$y(0) = (-20e^{-t/2} + C)\Big|_{t=0} = -20 + C = 4 \implies C = 24.$$

Therefore, the solution of the initial value problem is $y = -20e^{-t/2} + 24$. $\square$

Proposition 1.1 (Solution of a First-Order Linear Differential Equation)**.** The general solution of the first-order equation $y'(t) = ky + b$, where k and b are specified real numbers, is $y = Ce^{kt} - b/k$, where C is an arbitrary constant. Given an initial condition, the value of C may be determined.

Proof. Given that $y'(t) = ky + b$, we begin by dividing both sides of the equation $y'(t) = ky + b$ by $ky + b$, which gives

$$\frac{y'(t)}{ky + b} = 1.$$

Integrating both sides od this equation with respect to t,

$$\int \frac{y'(t)}{ky + b}\,dt = \int dt \iff \int \frac{1}{ky + b}\,dy = \int dt.$$

Using change of variable (u-substitution) we have

$$\frac{1}{k}\ln|ky + b| = t + D.$$

Assume that $ky + b \geq 0$, solving for y gives

$$y = e^{kD+kt} - \frac{b}{k} = e^{kD}e^{kt} - \frac{b}{k} = Ce^{kt} - \frac{b}{k},$$

where $C = e^{kD}$ is a constant. $\square$