MATH 2205 - Calculus II Lecture Notes 12

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1 Integration Techniques

1.1 Review of Partial Fraction Decomposition

Proposition 1.1 (Partial Fraction Decomposition). Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

(a) *Simple linear factor.* A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.

(b) *Repeated linear factor.* A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

(c) *Simple irreducible quadratic factor.* An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$

(d) *Repeated irreducible quadratic factor.* An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

1.2 Numerical Integration (Bonus Problem Topic)

Definition 1.1 (Absolute and Relative Error). Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common meaasures of the error in c as an approximation to x:
$$\textbf{absolute error} = |c - x|$$
and
$$\textbf{relative error} = \frac{c-x}{x}, \text{(if } x \neq 0).$$

Example 1.1. The ancient Greeks used $\dfrac{22}{7}$ to approximate the value of π. Determine the aboslute and relative error in this approximation to π.

Definition 1.2. Suppose f is defined an integrable on $[a, b]$. The *Midpoint Rule approximation* to $\displaystyle\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is
$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x,$$
where $\Delta x = (b-a)/n$, $x_0 = a$, $x_k = a + k\Delta x$, and $m_k = (x_{k-1}+x_k)/2 = a + (k-1/2)\Delta x$ is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, \ldots, n$.

Definition 1.3 (Trapezoid Rule)**.** Suppose f is defined and integrable on $[a, b]$. The *Trapezoid Rule approximation* to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x.$$

where $\Delta x = (b-a)/n$ and $x_k = a + k\Delta x$, for $k = 0, 1, 2, \ldots, n$.

Definition 1.4 (Simpson's Rule)**.** Suppose f is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The *Simpson's Rule approximation* to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$\begin{aligned} S(n) &= [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]\frac{\Delta x}{3} \\ &= \sum_{k=0}^{n/2-1} [f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})]\frac{\Delta x}{3}. \end{aligned}$$

where n is an even integer, $\Delta x = (b-a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

Theorem 1.1 (Errors in Numerical Integration)**.** Assume that f'' is continuous on the interval $[a, b]$ and that k is a real number such that $|f''(x)| \leq k$, for all x in $[a, b]$. The absolute errors in approximating the integral $\int_a^b f(x)\,dx$ by the Midpoint Rule and Trapezoid Rule with n subintervals satisfy the inequalities

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \text{ and } E_T \leq \frac{k(b-a)}{12}(\Delta x)^2,$$

respectively, where $\Delta x = (b-a)/n$.
Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a real number such that $|f^{(4)}(x)| \leq K$ on $[a, b]$. The absolute error in approximating the integral $\int_a^b f(x)\,dx$ by Simpson's Rule with n subintervals satisfies the inequality

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4.$$

Example 1.2. Approximate $\int_2^4 x^2\,dx$ using the Midpoint Rule/Trapezoid Rule/Simpson's Rule with $n = 4$ subintervals.

Solution. First, we split the interval $[a, b] = [2, 4]$ into $n = 4$ subintervals, of which the length is $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$. Then $x_k = a + k\Delta x = 2 + \frac{k}{2}$.

- *Midpoint Rule*: the midpoint of kth subinterval is $m_k = (x_{k-1} + x_k)/2 = a + (k - 1/2)\Delta x =$

$2 + \frac{k-1/2}{2}$. Then the Midpoint Rule approximation is

$$\begin{aligned}
\sum_{k=1}^{n} f(m_k)\Delta x &= \sum_{k=1}^{n} f\left(2 + \frac{k-1/2}{2}\right)\frac{1}{2} \\
&= \frac{1}{2}\sum_{k=1}^{n}\left(2 + \frac{k-1/2}{2}\right)^2 \\
&= \frac{1}{2}\sum_{k=1}^{n}\left[4 + (2k-1) + \frac{k^2 - k + 1/4}{4}\right] \\
&= \frac{1}{8}\sum_{k=1}^{n}\left(k^2 + 7k + \frac{49}{4}\right) \\
&= \frac{1}{8}\left(\sum_{k=1}^{n} k^2 + 7\sum_{k=1}^{n} k + \sum_{k=1}^{n}\frac{49}{4}\right) \\
&= \frac{1}{8}\left(\frac{n(n+1)(2n+1)}{6} + 7 \times \frac{n(n+1)}{2} + \frac{49}{4} \times n\right)\Bigg|_{n=4} \\
&= \frac{1}{8}\left(\frac{4(4+1)(2\times 4+1)}{6} + 7 \times \frac{4(4+1)}{2} + \frac{49}{4} \times 4\right) \\
&= \frac{1}{8}(30 + 70 + 49) \\
&= \frac{149}{8}.
\end{aligned}$$

- *Trazepoid Rule*:

$$\begin{aligned}
T(n) &= \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x \\
&= \left[\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f\left(2 + \frac{k}{2}\right) + \frac{1}{2}f(b)\right]\Delta x \\
&= \left[2 + \sum_{k=1}^{n-1}\left(2^2 + 2k + \frac{k^2}{4}\right) + 8\right]\frac{1}{2} \\
&= \left[2 + \frac{1}{4}\sum_{k=1}^{n-1}\left(16 + 8k + k^2\right) + 8\right]\frac{1}{2} \\
&= \left[2 + \frac{1}{4}\left(\sum_{k=1}^{n-1} 16 + 8\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} k^2\right) + 8\right]\frac{1}{2} \\
&= \left[2 + \frac{1}{4}\left(16 \times n + 8 \times \frac{n(n+1)}{2} + \frac{n(n+1)\times(2n+1)}{6}\right)\Bigg|_{n=3} + 8\right]\frac{1}{2} \\
&= \frac{150}{8}.
\end{aligned}$$

- *Simpson's Rule*:

$$
\begin{aligned}
S_n &= \sum_{k=0}^{n/2-1} [f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})]\frac{\Delta x}{3} \\
&= \sum_{k=0}^{n/2-1} [f(2+k) + 4f(2+k+1/2) + f(2+k+1)]\frac{1/2}{3} \\
&= \sum_{k=0}^{n/2-1} [(2+k)^2 + 4(5/2+k)^2 + (3+k)^2]\frac{1/2}{3} \\
&= \sum_{k=0}^{n/2-1} [(2^2+2k+k^2) + (25+20k+4k^2) + (3^3+6k+k^2)]\frac{1}{6} \\
&= \sum_{k=0}^{n/2-1} (6k^2+28k+38)\frac{1}{6} \\
&= \frac{1}{6}\left(6\sum_{k=0}^{n/2-1} k^2 + 28\sum_{k=0}^{n/2-1} k + 38\sum_{k=0}^{n/2-1} 1\right) \\
&= \frac{1}{6}\left(6\sum_{k=1}^{n/2-1} k^2 + 28\sum_{k=1}^{n/2-1} k + 38\sum_{k=1}^{n/2-1} 1 + 38\right) \\
&= \frac{1}{6}\left[6 \times \frac{n(n+1)(2n+1)}{6} + 28 \times \frac{n(n+1)}{2} + 38n + 38\right]\Bigg|_{n=4/2-1=1} \\
&= \frac{110}{6}.
\end{aligned}
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