MATH 2205 - Calculus II Lecture Notes 09

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1 Integration Techniques

1.1 Basic Approaches

Proposition 1.1 (Basic Integration Formulas)**.**

(a) $\int k\,dx = kx + C$, $k \in \mathbb{R}$ (k is real).

(b) $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$, $p \neq -1 \in \mathbb{R}$.

(c) $\int \cos ax\,dx = \dfrac{1}{a}\sin ax + C$.

(d) $\int \sin ax\,dx = -\dfrac{1}{a}\cos ax + C$.

(e) $\int \sec^2 ax\,dx = \dfrac{1}{a}\tan ax + C$.

(f) $\int \csc^2 ax\,dx = -\dfrac{1}{a}\cot ax + C$.

(g) $\int \sec ax \tan ax\,dx = \dfrac{1}{a}\sec ax + C$.

(h) $\int \csc ax \cot ax\,dx = -\dfrac{1}{a}\csc ax + C$.

(i) $\int e^{ax}\,dx = \dfrac{1}{a}e^{ax} + C$.

(j) $\int \dfrac{1}{x}\,dx = \ln|x| + C$.

(k) $\int \dfrac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\dfrac{x}{a} + C$.

(l) $\int \dfrac{1}{a^2 + x^2}\,dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$.

(m) $\int \dfrac{1}{x\sqrt{x^2 - a^2}}\,dx = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$, $a > 0$.

1.2 Integration by Parts

Theorem 1.1 (Integration by Parts)**.** Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v du.$$

Proof. By product rule, we have

$$\frac{d[u(x)v(x)]}{dx} = u'(x)v(x) + u(x)v'(x).$$

By integrating both sides, we can write this rule in terms of an indefinite integral:

$$u(x)v(x) = \int [u'(x)v(x) + u(x)v'(x)]\, dx$$

Rearranging this express in the form

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int v(x)u'(x)\, dx,$$

leads to the basic relationship for *integration by parts*. It is expressed compactly by noting that $du = u'(x)\, dx$ and $dv = v'(x)dx$. Suppressing the independent variable x, we have

$$\int udv = uv - \int vdu.$$

□

Example 1.1. Evaluate the following integrals.

(a) $\int xe^x\, dx$.

(b) $\int x\sin x\, dx$.

(c) $\int x^2e^x\, dx$.

(d) $\int x^ne^x\, dx$, where n is a positive integer.

(e) $\int x^n\sin x\, dx$.

(f) $\int \sin xe^{2x}\, dx$.

SOLUTION.

(a)

$$\begin{aligned}\int xe^x\, dx &= \int xde^x && [u = x, dv = de^x \implies v = e^x]\\ &= xe^x - \int e^x\, dx && [\text{Integration by parts } \int u\, dv = uv - \int v\, du]\\ &= xe^x - e^x + C.\end{aligned}$$

(b)

$$\begin{aligned}\int x\sin x\, dx &= \int x\, d(-\cos x) && [u = x, dv = d(-\cos x) \implies v = -\cos x]\\ &= x(-\cos x) - \int(-\cos x)\, dx && [\int u\, dv = uv - \int v\, du]\\ &= -x\cos x + \int \cos x\, dx\\ &= -x\cos x + \sin x + C.\end{aligned}$$

(c)

$$
\begin{aligned}
\int x^2 e^x\,dx &= \int x^2 de^x && [u = x^2, dv = de^x \implies v = e^x] \\
&= x^2 e^x - \int e^x\,dx^2 && [\int u\,dv = uv - \int v\,du] \\
&= x^2 e^x - \int e^x 2x\,dx \\
&= x^2 e^x - 2\int e^x x\,dx \\
&= x^2 e^x - 2\int x\,de^x && [u = x, dv = de^x \implies v = e^x] \\
&= x^2 e^x - 2\left(xe^x - \int e^x\,dx\right) && [\int udv = uv - \int vdu] \\
&= x^2 e^x - 2(xe^x - e^x) + C.
\end{aligned}
$$

(d) For this integral, we need to apply integration by parts multiple times.

$$
\begin{aligned}
\int x^n e^x\,dx &= \int x^n\,de^x \qquad [u = x^n, dv = de^x \implies v = e^x] \\
&= x^n e^x - \int e^x\,dx^n \qquad [\int u\,dv = uv - \int v\,du] \\
&= x^n e^x - \int e^x n x^{n-1}\,dx \\
&= x^n e^x - n\int x^{n-1} e^x\,dx \\
&= x^n e^x - n\int x^{n-1}\,de^x \\
&= x^n e^x - n\left(e^x x^{n-1} - \int e^x\,dx^{n-1}\right) \\
&= x^n e^x - n\left[e^x x^{n-1} - \int e^x\,(n-1)x^{n-2}dx\right] \\
&= x^n e^x - n\left[e^x x^{n-1} - (n-1)\int x^{n-2} e^x\,dx\right] \\
&= x^n e^x - ne^x x^{n-1} + n(n-1)\int x^{n-2} e^x\,dx \\
&= x^n e^x - ne^x x^{n-1} + n(n-1)\left[x^{n-2}e^x - (n-2)\int x^{n-3} e^x\,dx\right] \\
&= x^n e^x - ne^x x^{n-1} + n(n-1)x^{n-2}e^x - n(n-1)(n-2)\int x^{n-3} e^x\,dx \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} x^{n-k} e^x + C,
\end{aligned}
$$

where $n! = n(n-1)(n-2)\cdots 2 \times 1$.

(e)

$$
\begin{aligned}
\int x^n \sin x \, dx = & \int x^n \, d(-\cos x) \\
= & \; x^n(-\cos x) - \int (-\cos x) \, dx^n \\
= & -x^n \cos x + n \int \cos x x^{n-1} \, dx \\
= & -x^n \cos x + n \int x^{n-1} \, d \sin x \\
= & -x^n \cos x + n \left(x^{n-1} \sin x - \int \sin x \, dx^{n-1} \right) \\
= & -x^n \cos x + n \left[x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x \, dx \right] \\
= & -x^n \cos x + n x^{n-1} \sin x + n(n-1) \int x^{n-2} \sin x \, dx \\
= & -x^n \cos x + n x^{n-1} \sin x \\
& + n(n-1) \left[-x^{n-2} \cos x + (n-2) x^{n-3} \sin x + (n-2)(n-3) \int x^{n-4} \sin x \, dx \right] \\
= & -x^n \cos x + n x^{n-1} \sin x - n(n-1) x^{n-2} \cos x + n(n-1)(n-2) x^{n-3} \sin x \\
& + n(n-1)(n-2)(n-3) \int x^{n-4} \sin x \, dx \\
= & \sum_{k=0}^{n} (-1)^{k+1} \frac{n!}{(n-k)!} x^{n-k} \left[\frac{1+(-1)^k}{2} \cos x + \frac{1-(-1)^k}{2} \sin x \right] + C.
\end{aligned}
$$

(f)

$$
\begin{aligned}
\int \sin x e^{2x} \, dx &= \int e^{2x} \, d(-\cos x) \\
&= e^{2x}(-\cos x) - \int (-\cos x) \, de^{2x} \\
&= -e^{2x} \cos x + 2 \int \cos x e^{2x} \, dx \\
&= -e^{2x} \cos x + 2 \int e^{2x} \, d \sin x \\
&= -e^{2x} \cos x + 2 \left(e^{2x} \sin x - \int \sin x \, de^{2x} \right) \\
&= -e^{2x} \cos x + 2 \left(e^{2x} \sin x - 2 \int \sin x e^{2x} \, dx \right) \\
&= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int \sin x e^{2x} \, dx. \\
&= e^{2x}(2 \sin x - \cos x) - 4 \int \sin x e^{2x} \, dx.
\end{aligned}
$$

Observe that $\int \sin x e^{2x} \, dx$ appears again. We can collect the term $\int \sin x e^{2x}$, and then solve for it.

$$5\int \sin x e^{2x}\,dx = e^{2x}(2\sin x - \cos x) \implies \int \sin x e^{2x}\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C.$$

□

Theorem 1.2 (Integration by Parts for Definite Integrals)**.** Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

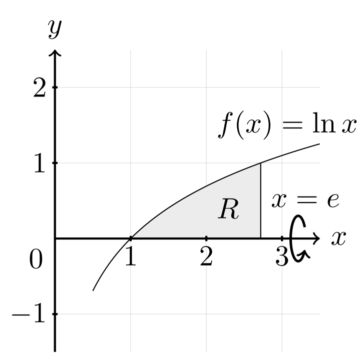
Example 1.2. Evaluate $\displaystyle\int_1^2 \ln x\,dx$.

Solution.

$$\begin{aligned}\int_1^2 \ln x\,dx &= x\ln x\Big|_1^2 - \int_1^2 x\,d\ln x\\ &= (2\ln 2 - 1\ln 1) - \int_1^2 x\frac{1}{x}\,dx\\ &= 2\ln 2 - \int_1^2 1\,dx\\ &= 2\ln 2 - x\Big|_1^2\\ &= 2\ln 2 - (2-1)\\ &= 2\ln 2 - 1.\end{aligned}$$

□

Example 1.3 (Solids of revolution)**.** Let R be the region bounded by $y = \ln x$, the x-axis, and the line $x = e$. Find the volume of the solid that is generated when the region R is revolved about the x-axis.



Solution. Rovlving R about the x-axis generates a solid whose volume is computed with the disk method. Its volume is

$$\begin{aligned}
V &= \int_1^e \pi(\ln x)^2\,dx \\
&= \pi \int_1^e (\ln x)^2\,dx && [u = x, dv = d(\ln x)^n \implies v = (\ln x)^2] \\
&= \pi \left[x(\ln x)^2 \Big|_1^e - \int_1^e x\,d(\ln x)^2 \right] && \text{[Integration by parts]} \\
&= \pi \left[e - \int_1^e x 2\ln x \frac{1}{x}\,dx \right] \\
&= \pi \left(e - 2\int_1^e \ln x\,dx \right) \\
&= \pi e - 2\pi \int_1^e \ln x\,dx \\
&= \pi e - 2\pi \left(x\ln x \Big|_1^e - \int_1^e x\,d\ln x \right) \\
&= \pi e - 2\pi \left(e - \int_1^e x\frac{1}{x}\,dx \right) \\
&= \pi e - 2\pi \left(e - x\Big|_1^e \right) \\
&= \pi e - 2\pi\,[e - (e-1)] \\
&= \pi e - 2\pi \\
&= \pi(e-2).
\end{aligned}$$

□