MATH 2205 - Calculus II Lecture Notes 08

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1 Integration

1.1 Sigma (Summation) Notation

Proposition 1.1.

- *Constant Multiple Rule.*
$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k.$$
- *Addition Rule.*
$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

Theorem 1.1 (Sums of Powers of Integers)**.** Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

1.2 Approximating Areas under Curves

Definition 1.1 (Riemann Sum)**.** Suppose f is a function defined on a closed interval $[a,b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

is called a *Riemann sum* for f on $[a,b]$. This sum is called

- a *left Riemann sum* if x_k^* is the left endpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = x_{k-1} = a + (k-1)\Delta x$;
- a *right Riemann sum* if x_k^* is the right endpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = x_k = a + k\Delta x$; and
- a *midpoint Riemann sum* if x_k^* is the midpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = (x_{k-1} + x_k)/2 = a + (k - 1/2)\Delta x$, for $k = 1, 2, \ldots, n$.

1.3 Definite Integrals

Definition 1.2 (Net Area)**.** Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The *net area* of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a,b]$.

Definition 1.3 (Definite Integral)**.** A function f defined on $[a,b]$ is *integrable* on $[a,b]$ if

$$\lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and is unique over all partitions of $[a,b]$ and all choices of x_k^* on a partition. This limit is the *definite integral* of f from a to b, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

To simplify the calculation, we use equally spaced grid points and right Riemann sums. That is, for each value of n, we let $\Delta x_k = \Delta x = (b-a)/n$ and $x_k^* = a + k\Delta x$, for $k = 1, 2, \ldots, n$. Then $n \to \infty$ and $\Delta \to 0$,

$$\int_a^b f(x)\,dx = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to\infty}\sum_{k=1}^{n} f(a+k\Delta x)\Delta x.$$

Definition 1.4 (Integrable Functions)**.** If f is continuous on $[a,b]$ or bounded on $[a,b]$ with a finite number of discontinuities, then f is integrable on $[a,b]$.

Definition 1.5 (Reversing Limits and Identical Limits of Integration)**.** Suppose f is integrable on $[a,b]$.

(a) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

(b) $\displaystyle\int_a^a f(x)\,dx = 0.$

Proposition 1.2 (Properties of Definite Integrals)**.**

(a) *Integral of a Sum.* Assume f and g are integrable on $[a,b]$, then their sum $f+g$ is integrable on $[a,b]$ and the integral of their sum is the sum of their integrals:

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(b) *Constants in Integrals.* If f is integrable on $[a,b]$ and c is a constant, then cf is integrable on $[a,b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

(c) *Integrals over Subintervals.* If the point p is distinct from a and b, then the integral on $[a,b]$ may be split into two integrals,

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx.$$

1.4 Fundamental Theorem of Calculus

Definition 1.6 (Area Function)**.** Let f be a continuous function, for $t \geq a$. The *area function* for f with left endpoint a is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Theorem 1.2 (Fundamental Theorem of Calculus (FTOC), Part I)**.** If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b.$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Theorem 1.3 (Fundamental Theorem of Calculus (FTOC), Part II)**.** If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

Proposition 1.3 (Antiderivatives)**.**

(a) $\displaystyle\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.

(b) $\displaystyle\int x^{-1}\,dx = \ln|x| + C$.

(c) $\displaystyle\int e^x\,dx = e^x + C$.

(d) $\displaystyle\int \sin x\,dx = -\cos x + C$.

(e) $\displaystyle\int \cos x\,dx = \sin x + C$.

(f) $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$.

(g) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$.

(h) $\displaystyle\int \sec x \tan x\,dx = \sec x + C$.

(i) $\displaystyle\int \sec^2 x\,dx = \tan x + C$.

1.5 Working with Integrals

Theorem 1.4 (Integrals of Even and Odd Functions)**.** Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, i.e., $f(-x) = f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, i.e., $f(-x) = -f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Definition 1.7 (Average Value of a Function)**.** The average value of an integrable function f on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx.$$

Theorem 1.5 (Mean Value Theorem for Integrals)**.** Let f continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dt.$$

1.6 Substitution Rule

Theorem 1.6 (Substitution Rule for Indefinite Integrals)**.** Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Procedure 1.1 (Substitution Rule (Change of Variables))**.**

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Theorem 1.7 (Substitution Rule for Definite Integrals)**.** Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proposition 1.4 (Properties of Trig Functions)**.**

(a) $\sin^2\theta + \cos^2\theta = 1$.

(b) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

(c) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

(d) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

(e) $\sin 2\theta = 2\sin\theta\cos\theta$.

(f) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.

(g) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

(h) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

(i) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

2 Application of Integration

2.1 Regions Between Curves

Definition 2.1 (Approximation of Area of a Region Between Curves using Riemman Sum)**.** Suppose that f and g continuous on an interval $[a, b]$ on which $f(x) \geq g(x)$. Partition the interval $[a, b]$ into n subintervals using uniformly spaced grid points separated by a distance $\Delta x = (b - a)/n$. Then the area A of the region bounded by the two curves and the vertical lines $x = a$ and $x = b$ can be approximated by:

$$A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x,$$

where $x_k^* \in [x_{k-1}, x_k], k = 1, 2, \ldots, n, x_0 = a, x_n = b$.

Definition 2.2 (Area of a Region Between Two Curves)**.** Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x.$$

Definition 2.3 (Area of a Region Between Two Curves with Respect to y)**.** Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

2.2 Volume by Slicing

Definition 2.4 (General Slicing Method)**.** Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. Then volume of the solid is

$$V = \int_a^b A(x)\, dx.$$

Definition 2.5 (Disk Method about the x-Axis)**.** Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2\, dx.$$

Definition 2.6 (Washer Method about the x-Axis)**.** Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi [f(x)^2 - g(x)^2]\, dx.$$

Definition 2.7 (Disk and Washer Methods about the y-Axis)**.** Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi[p(y)^2 - q(y)^2]\, dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2\, dy.$$

2.3 Volume by Shells

Definition 2.8. Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x[f(x) - g(x)]\, dx.$$

2.4 Length of Curves

Definition 2.9 (Arc Length for $y = f(x)$)**.** Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Definition 2.10 (Arc Length for $x = g(y)$)**.** Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. Then length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

2.5 Surface Area

Definition 2.11 (Area of a Surface of Revolution)**.** Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

2.6 Physical Applications

Definition 2.12 (Mass of a One-Dimensional Object)**.** Suppose a thin bar or wire is represented by the interval $a \leq x \leq b$ with a density function ρ (with units of mass per length). The *mass* of the object is

$$m = \int_a^b \rho(x)\, dx.$$

Definition 2.13 (Work)**.** The work done by a variable force F moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx.$$

Theorem 2.1 (Hooke's law)**.** The force required to keep the spring in a compressed or stretched position x units from the equilibrium position is

$$F(x) = kx,$$

where the positive spring constant k measures the stiffness of the spring. Note that to stretch the spring to a position $x > 0$, a force $F > 0$ (in the negative position) is required. To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required.

3 Algebra

3.1 Exponents and Radicals

(a) $\dfrac{1}{x^a} = x^{-a}$.
(b) $\sqrt[n]{x} = x^{1/n}$.
(c) $x^{a+b} = x^a x^b$.
(d) $x^{a-b} = \dfrac{x^a}{x^b}$.
(e) $x^{ab} = (x^a)^b$.
(f) $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.
(g) $(xy)^a = x^a y^a$.
(h) $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$.

3.2 Logarithm

(a) $y = a^x \implies x = \log_a y$.
(b) $\log_e x = \ln x$.
(c) $\log_b(xy) = \log_b x + \log_b y$.
(d) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$.
(e) $\log_b(x^p) = p \log_b x$.
(f) $\log_b(x^{1/p}) = \dfrac{1}{p} \log_b x$.
(g) $\log_b x = \dfrac{\log_k x}{\log_k b}$.

3.3 Factoring Formulas

(a) $a^2 - b^2 = (a-b)(a+b)$.
(b) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
(c) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$.

3.4 Binomials

(a) $(a \pm b)^2 = a^2 \pm 2ab + b^2$.
(b) $(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$.

3.5 Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$