MATH 2205 - Calculus II Lecture Notes 07

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1 Application of Integration

1.1 Review of Shell Method, Length of Curves

Definition 1.1. Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x[f(x) - g(x)]\, dx.$$

Definition 1.2 (Arc Length for $y = f(x)$)**.** Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Definition 1.3 (Arc Length for $x = g(y)$)**.** Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. Then length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Example 1.1 (Arc length of an exponential curve)**.** Find the length of the curve $f(x) = 2e^x + \frac{1}{8}e^{-x}$ on the interval $[0, \ln 2]$.

Solution. By definition, we have

$$\begin{aligned}
L &= \int_0^{\ln 2} \sqrt{1 + f'(x)^2}\, dx \\
&= \int_0^{\ln 2} \sqrt{1 + \left(2e^x - \frac{1}{8}e^{-x}\right)^2}\, dx \\
&= \int_0^{\ln 2} \sqrt{1 + 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}}\, dx \\
&= \int_0^{\ln 2} \sqrt{4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}}\, dx \\
&= \int_0^{\ln 2} \sqrt{\left(2e^x + \frac{1}{8}e^{-x}\right)^2}\, dx \\
&= \int_0^{\ln 2} 2e^x + \frac{1}{8}e^{-x}\, dx \\
&= 2\int_0^{\ln 2} e^x\, dx + \frac{1}{8}\int_0^{\ln 2} e^{-x}(-1)(-1)\, dx && [\text{Substitute } u = -x, du = -dx] \\
&= 2e^x \Big|_0^{\ln 2} - \frac{1}{8}\int_0^{-\ln 2} e^u\, du \\
&= 2(e^{\ln 2} - e^0) - \frac{1}{8}e^u \Big|_0^{-\ln 2} \\
&= 2(2-1) - \frac{1}{8}(e^{-\ln 2} - e^0) \\
&= 2 - \frac{1}{8}\left(\frac{1}{2} - 1\right) \\
&= \frac{33}{16}.
\end{aligned}$$

□

1.2 Surface Area

Definition 1.4 (Area of a Surface of Revolution)**.** Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

Example 1.2 (Using the surface area formula)**.** The graph of $f(x) = 2\sqrt{x}$ on the interval $[1, 3]$ is revolved about the x-axis. What is the area of the surface generated?

Solution. By definition, the area of the surface of the region revolving about the x-axis (Figure

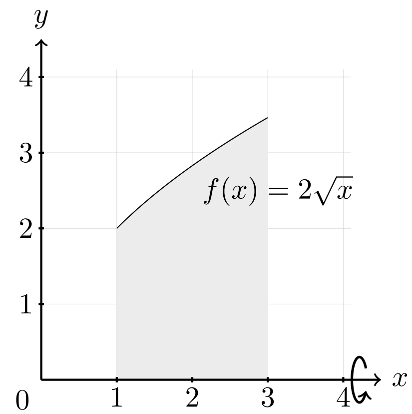

Figure 1: $f(x) = 2\sqrt{x}$

1) is

$$
\begin{aligned}
A &= \int_1^3 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \\
&= \int_1^3 4\pi\sqrt{x}\sqrt{1+1/x}\,dx \\
&= 4\pi\int_1^3 \sqrt{\underbrace{1+x}_{u}}\,\underbrace{dx}_{du} \\
&= 4\pi\int_{u(1)}^{u(3)} \sqrt{u}\,du \\
&= 4\pi\frac{2x^{3/2}}{3}\bigg|_2^4 \\
&= \frac{16\pi}{3}\left(4-\sqrt{2}\right).
\end{aligned}
$$

□

1.3 Physical Applications

Definition 1.5 (Mass of a One-Dimensional Object)**.** Suppose a thin bar or wire is represented by the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The *mass* of the object is

$$m = \int_a^b \rho(x)\,dx.$$

Example 1.3 (Mass from variable density)**.** A thin 2-m bar, represented by the interval $0 \le x \le 2$, is made of an alloy whose density in units of kg/m is given by $\rho(x) = (1+x^2)$. What is the mass of the bar?

Solution.

$$\begin{aligned} m &= \int_0^2 \rho(x)\,dx \\ &= \int_0^2 (1+x^2)\,dx \\ &= x + \frac{x^3}{3}\bigg|_0^2 \\ &= 2 + \frac{2^3}{3} \\ &= \frac{14}{3}. \end{aligned}$$

□

Definition 1.6 (Work)**.** The work done by a variable force F moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx.$$

Theorem 1.1 (Hooke's law)**.** The force required to keep the spring in a compressed or stretched position x units from the equilibrium position is $F(x) = kx$, where the positive spring constant k measures the stiffness of the spring. Note that to stretch the spring to a position $x > 0$, a force $F > 0$ (in the negative position) is required. To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required.

Example 1.4 (Compressing a spring)**.** Suppose a force of 10 N is required to stretch a spring 0.1 m from its equilibrium position and hold it in that position.

(a) Assuming that the spring obeys Hooke's law, find the spring constant k.
(b) How much work is needed to *compress* the spring 0.5 m from its equilibrium position?
(c) How much work is needed to *stretch* the spring 0.25 m from its equilibrium position?
(d) How much additional work is required to stretch the spring 0.25 m if it has already been stretched 0.1 m from its equilibrium position?

Solution.

(a) Given that a force of 10 N is required to stretch a spring 0.1 m from its equilibrium position, we can find the spring constant k as follows

$$F = kx \implies k = \frac{F}{x} = \frac{10}{0.1} = 100.$$

(b) The work is

$$W = \int_0^{-0.5} F(x)\,dx = \int_0^{-0.5} kx\,dx = \int_0^{-0.5} 100x\,dx = 50x^2\bigg|_0^{0.5} = 12.5.$$

(c) The work is

$$W = \int_0^{0.25} F(x)\,dx = \int_0^{0.25} kx\,dx = \int_0^{0.25} 100x\,dx = 50x^2\bigg|_0^{0.25} = \frac{25}{8}.$$

(d) The additional work is

$$W = \int_{0.1}^{0.25} F(x)\,dx = \int_{0.1}^{0.25} kx\,dx = \int_{0.1}^{0.25} 100x\,dx = 50x^2\Big|_{0.1}^{0.25} = \frac{25}{8} - \frac{1}{2} = \frac{21}{8}.$$

$\square$