MATH 2205 - Calculus II Lecture Notes 05

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1 Integration

1.1 Reivew of Properties of Trig Functions, Regions Between Curves

Proposition 1.1 (Properties of Trig Functions)**.**

(a) $\sin^2\theta + \cos^2\theta = 1$.

(b) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

(c) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

Definition 1.1 (Area of a Region Between Two Curves)**.** Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x.$$

1.2 Regions Between Curves (Continued)

Example 1.1 (Compound region)**.** Find the area of the region bounded by the graphs of $f(x) = -x^2 + 3x + 6$ and $g(x) = |2x|$ on the interval on which $f(x) \geq g(x)$.

Solution. The lower boundary of the region is bounded by two different branches of the absolute value function. Then we need to divide the region into two (or more) subregions whose areas are found independently and then summed. By the definition of absolute value,

$$g(x) = |2x| = \begin{cases} g_-(x) = -2x & \text{if } x < 0, \\ g_+(x) = 2x & \text{if } x \geq 0. \end{cases}$$

Next let us find the corresponding interval by solving $f(x) = g(x)$,

$$\begin{aligned}
\begin{cases} -x^2 + 3x + 6 = -2x & \text{if } x < 0, \\ -x^2 + 3x + 6 = 2x & \text{if } x \geq 0, \end{cases}
&\implies \begin{cases} -x^2 + 5x + 6 = (-x+6)(x+1) = 0 & \text{if } x < 0, \\ -x^2 + 1x + 6 = (-x+3)(x+2) = 0 & \text{if } x \geq 0, \end{cases} \\
&\implies \begin{cases} x = 6 \text{ (does not satisfy } x < 0\text{, ruled out)}, x = -1, \\ x = 3, x = -2 \text{ (does not satisfy } x \geq 0\text{, ruled out)}, \end{cases} \\
&\implies [a, b] = [-1, 3].
\end{aligned}$$

Then the intervals for two subregions are $[-1,0]$ and $[0,3]$. Then the area A of the region is

$$
\begin{aligned}
A &= \int_{-1}^{0} [f(x) - g_{-}(x)]\,dx + \int_{0}^{3} [f(x) + g_{+}(x)]\,dx \\
&= \int_{-1}^{0} [(-x^2+3x+6) - (-2x)]\,dx + \int_{0}^{3} [(-x^2+3x+6) - (2x)]\,dx \\
&= \int_{-1}^{0} -x^2+5x+6\,dx + \int_{0}^{3} -x^2+x+6\,dx \\
&= \left(-\frac{x^3}{3} + \frac{5x^2}{2} + 6x\right)\Bigg|_{-1}^{0} + \left(-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right)\Bigg|_{0}^{3} \\
&= \left(-\frac{1}{3} - \frac{5}{2} + 6\right) + \left(-\frac{3^3}{3} + \frac{3^2}{2} + 18\right) \\
&= \frac{19}{6} + \frac{27}{2} \\
&= \frac{50}{3}.
\end{aligned}
$$

□

Example 1.2 (Integrating with respect to x)**.** Find the area of the region R bounded by the graphs of $y = 4x$, $y = x+6$, and the x-axis.

Solution. The area of this region could be found by integrating with respect to x, which we need to split the region into two pieces. Let $f(x) = x+6$, $g(x) = 4x$, $h(x) = 0$. Hence we can find the subintervals by solving $f(x) = g(x)$, $f(x) = h(x)$, $g(x) = h(x)$:

$$
\begin{cases} f(x) = g(x) \\ f(x) = h(x) \\ g(x) = h(x) \end{cases} \implies \begin{cases} x+6 = 4x \\ x+6 = 0 \\ 4x = 0 \end{cases} \implies \begin{cases} x = 2, \\ x = -6, \\ x = 0. \end{cases}
$$

Then subintervals are $[-6,0]$ and $[0,2]$. Therefore, the area A of the region is

$$
\begin{aligned}
A &= \int_{-6}^{0} [f(x) - h(x)]\,dx + \int_{0}^{2} [f(x) - g(x)]\,dx \\
&= \int_{-6}^{0} [(x+6) - 0]\,dx + \int_{0}^{2} [(x+6) - 4x]\,dx \\
&= \int_{-6}^{0} x+6\,dx + \int_{0}^{2} 6-3x\,dx \\
&= \left(\frac{x^2}{2} + 6x\right)\Bigg|_{-6}^{0} + \left(6x - \frac{3x^2}{2}\right)\Bigg|_{0}^{2} \\
&= 0 - \left(\frac{(-6)^2}{2} + 6\times(-6)\right) + \left(6\times 2 - \frac{3\times 2^2}{2}\right) - 0 \\
&= 18 + 6 \\
&= 24.
\end{aligned}
$$

□

Definition 1.2 (Area of a Region Between Two Curves with Respect to y)**.** Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

Example 1.3 (Integrating with respect to y)**.** Find the area of the region R bounded by the graphs of $y = 4x$, $y = x + 6$, and the x-axis.

Solution. The area of this region could be found by integrating with respect to x. But this approach requires splitting the region into two pieces. Alternatively, we can view y as the independent variable, express the bounding curve as functions of y, and make horizontal slices parallel to the x-axis. Solving for x in terms of y,

$$\begin{cases} y = 4x \\ y = x + 6 \end{cases} \implies \begin{cases} x = f(y) = \dfrac{y}{4}, \\ x = g(y) = y - 6. \end{cases}$$

Note that the x-axis can be expressed as $y = 0$. Next let us find the other endpoint of the interval by solving $f(y) = g(y)$, that is,

$$\frac{y}{4} = y - 6 \implies \frac{3y}{4} = 6 \implies y = 8.$$

Hence, the area A of the region is given by

$$\begin{aligned} A &= \int_0^8 [f(y) - g(y)]\, dy \\ &= \int_0^8 [\frac{y}{4} - (y - 6)]\, dy \\ &= \int_0^8 6 - \frac{3y}{4}\, dy \\ &= 6y - \frac{3y^2}{8}\bigg|_0^8 && \text{[FTOC]} \\ &= 48 - \frac{3 \times 8^2}{8} \\ &= 24. \end{aligned}$$

□

1.3 Volume by Slicing

Definition 1.3 (General Slicing Method)**.** Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. Then volume of the solid is

$$V = \int_a^b A(x)\, dx.$$

Example 1.4 (Volume of a "parabolic cube")**.** Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has a base R, and cross sections through the solid perpendicular to the base and parallel to the y-axis are squares. Find the volume of the solid.

Solution. Focus on a cross section through the solid at a point x, where $0 \leq x \leq 1$. That cross section is a square with sides of length $1 - x^2$. Therefore, the area of a typical cross section is $A(x) = (1 - x^2)^2$. Using the general slicing method, the volume V of the solid is

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 (1-x^2)^2\,dx \\ &= \int_0^1 1 - 2x^2 + x^4\,dx \\ &= x - \frac{2x^3}{3} + \frac{x^5}{5}\bigg|_0^1 \\ &= 1 - \frac{2}{3} + \frac{1}{5} \\ &= \frac{8}{15}. \end{aligned}$$

□

Definition 1.4 (Disk Method about the x-Axis)**.** Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2\,dx.$$

Example 1.5 (Disk method at work)**.** Let R be the region bounded by the curve $f(x) = (x+1)^2$, the x-axis, and the lines $x = 0$ and $x = 2$. Find the volume of the solid of revolution obtained by revolving R about the x-axis.

Solution. When the region R is resolved about the x-axis, it generates a solid of revolution. A cross section perpendicular to the x-axis at the point $0 \leq x \leq 2$ is a circular disk radius $f(x)$. Therefore, a typical cross section has area

$$A(x) = \pi f(x)^2 = \pi[(x+1)^2]^2 = \pi(x+1)^4.$$

Integrating these cross-sectional areas between $x = 0$ and $x = 2$ gives the volume V of the solid:

$$\begin{aligned} V &= \int_0^2 A(x)\,dx \\ &= \int_0^2 \pi(x+1)^4\,dx \\ &= \int_1^3 \pi u^4\,du \\ &= \pi \frac{u^5}{5}\bigg|_1^3 \\ &= \frac{242\pi}{5}. \end{aligned}$$

$\square$

Definition 1.5 (Washer Method about the x-Axis)**.** Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]\,dx.$$

Example 1.6 (Volume by the washer method)**.** The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that result when R is revolved about the x-axis?

Solution. The region R is bounded by the graphs of f and g with $f(x) \geq g(x)$ on $[0, 1]$, so the washer method is applicable. The area of a typical cross section at the point x is

$$A(x) = \pi[f(x)^2 - g(x)^2] = \pi[x - x^4].$$

Therefore, the volume V of the solid is

$$\begin{aligned} V &= \int_0^1 \pi(x - x^4)\,dx \\ &= \pi \int_0^1 x - x^4\,dx \\ &= \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\bigg|_0^1 \\ &= \pi\left(\frac{1}{2} - \frac{1}{5}\right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

$\square$

Definition 1.6 (Disk and Washer Methods about the y-Axis)**.** Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines

$y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi[p(y)^2 - q(y)^2]\, dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2\, dy.$$

Example 1.7 (Which solid has greater volume?)**.** Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

Solution. Solving $y^3 = 4y \implies y(y^2 - 4) = 0$, so the intersections in the first quadrant are $y = 0$, $y = 2$. Hence the corresponding points are $(0, 0)$ and $(8, 2)$. Therefore, the volume V_x of the region R revolving about x-axis is

$$\begin{aligned} V_x &= \int_0^8 \pi \left[(x^{1/3})^2 - \left(\frac{x}{4}\right)^2 \right] dx \\ &= \pi \int_0^8 x^{2/3} - \frac{x^2}{16}\, dx \\ &= \pi \left(\frac{3x^{5/3}}{5} - \frac{x^3}{48} \right) \Bigg|_0^8 \\ &= \pi \left(\frac{38^{5/3}}{5} - \frac{8^3}{48} \right) \\ &= \pi \left(\frac{3 \times 32}{5} - \frac{32}{3} \right) \\ &= \frac{128\pi}{15}. \end{aligned}$$

On the other hand, the volume V_y of the region R revolving about y-axis is

$$\begin{aligned} V_y &= \int_0^2 \pi \left[(4y)^2 - (y^3)^2 \right] dy \\ &= \pi \int_0^2 16y^2 - y^6\, dy \\ &= \pi \left(\frac{16y^3}{3} - \frac{y^7}{7} \right) \Bigg|_0^2 \\ &= \pi \left(\frac{16 \times 2^3}{3} - \frac{2^7}{7} \right) \\ &= \pi \left(\frac{128}{3} - \frac{128}{7} \right) \\ &= \frac{512\pi}{21}. \end{aligned}$$

$\square$