MATH 2205 - Calculus II Lecture Notes 02

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1 Integration

1.1 Review of Sigma Notation, Riemann Sum, and Definite Integral

Proposition 1.1. Suppose that $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two sets of real numbers, and suppose that $c \in \mathbb{R}$ is a real number.

- *Constant Multiple Rule.* We can factor multiplicative constants out of sum:

$$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k.$$

- *Addition Rule.* We can also split a sum into two sums:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

Definition 1.1 (Definite Integral)**.** A function f defined on $[a, b]$ is *integrable* on $[a, b]$ if

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the *definite integral* of f from a to b, which we write

$$\int_a^b f(x)\, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k.$$

Definition 1.2 (Reversing Limits and Identical Limits of Integration)**.** Suppose f is integrable on $[a, b]$.

(a) $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx.$

(b) $\displaystyle\int_a^a f(x)\, dx = 0.$

Proposition 1.2 (Properties of Definite Integrals)**.**

(a) *Integral of a Sum.*

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

(b) *Constants in Integrals.*

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx.$$

(c) *Integrals over Subintervals.*

$$\int_a^b f(x)\, dx = \int_a^p f(x)\, dx + \int_p^b f(x)\, dx.$$

1.2 Fundamental Theorem of Calculus

Definition 1.3 (Area Function)**.** Let f be a continuous function, for $t \geq a$. The *area function* for f with left endpoint a is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Theorem 1.1 (Fundamental Theorem of Calculus (FTOC), Part I)**.** If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b.$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Proof.

$$\begin{aligned}
A'(x) = \frac{d}{dx}A(x) &= \lim_{h\to 0}\frac{A(x+h) - A(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt \\
&= \lim_{h\to 0}\frac{1}{h}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(x + k\Delta t)\Delta t\right) && \text{[using right Riemann sum]} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x + \frac{kh}{n}\right)\frac{h}{n}\right] && \text{[by } \Delta t = \frac{h}{n}\text{]} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\lim_{n\to\infty}\frac{h}{n}\sum_{k=1}^{n} f\left(x + \frac{kh}{n}\right)\right] \\
&= \lim_{n\to\infty}\lim_{h\to 0}\frac{1}{n}\sum_{k=1}^{n} f\left(x + \frac{kh}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\lim_{h\to 0} f\left(x + \frac{kh}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x) \\
&= \lim_{n\to\infty}\frac{1}{n}nf(x) \\
&= \lim_{n\to\infty} f(x) \\
&= f(x).
\end{aligned}$$

□

Theorem 1.2 (Fundamental Theorem of Calculus (FTOC), Part II)**.** If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

Proof. By FTOC Part I, the area function A is an antiderivative of f. Given that F is any antiderivative of f on $[a, b]$, we have

$$F(x) = A(x) + C,$$

where C is a constant. Note that $A(a) = \int_a^a f(x)\,dx = 0$, it follows that

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) + C - C = A(b) = \int_a^b f(x)\,dx.$$

□

Proposition 1.3 (Antiderivatives)**.**

(a) $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$, where $p \neq -1$.

(b) $\int x^{-1}\,dx = \ln|x| + C$.

(c) $\int e^x\,dx = e^x + C$.

(d) $\int \sin x\,dx = -\cos x + C$.

(e) $\int \cos x\,dx = \sin x + C$.

(f) $\int \dfrac{1}{1+x^2}\,dx = \arctan x + C$.

(g) $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$.

(h) $\int \sec x \tan x\,dx = \sec x + C$.

(i) $\int \sec^2 x\,dx = \tan x + C$.

Example 1.1 (Evaluating definite integrals)**.** Evaluate the following definite integrals using the Fundamental Theorem of Calculus. Interpret each result geometrically.

(a) $\int_0^{10} (60x - 6x^2)\,dx$.

(b) $\int_0^{2\pi} 3\sin x\,dx$.

(c) $\int_{1/16}^{1/4} \dfrac{\sqrt{t} - 1}{t}\,dt$.

SOLUTION.

(a)

$$\begin{aligned}
\int_0^{10} (60x - 6x^2)\,dx &= 60\int_0^{10} x\,dx - 6\int_0^{10} x^2\,dx \\
&= 60\left(\frac{x^2}{2} + C\right)\Bigg|_0^{10} - 6\left(\frac{x^3}{3} + C\right)\Bigg|_0^{10} \\
&= 60\left(\frac{10^2}{2} - 0\right) - 6\left(\frac{10^3}{3} - 0\right) \\
&= 3000 - 2000 \\
&= 1000.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{2\pi} 3\sin x\,dx &= 3\int_0^{2\pi} \sin x\,dx \\
&= 3(-\cos x + C)\Bigg|_0^{2\pi} \\
&= 3(-\cos(2\pi) - (-\cos(0))) \\
&= 3(-1 + 1) \\
&= 0.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t}\,dt &= \int_{1/16}^{1/4} \frac{\sqrt{t}}{t} - \frac{1}{t}\,dt \\
&= \int_{1/16}^{1/4} t^{-1/2} - t^{-1}\,dt \\
&= \int_{1/16}^{1/4} t^{-1/2}\,dt - \int_{1/16}^{1/4} t^{-1}\,dt \\
&= \frac{t^{1/2}}{1/2}\Bigg|_{1/16}^{1/4} - \ln|t|\Bigg|_{1/16}^{1/4} \\
&= 2t^{1/2}\Bigg|_{1/16}^{1/4} - \ln|t|\Bigg|_{1/16}^{1/4} \\
&= 2\left[\left(\frac{1}{4}\right)^{1/2} - \left(\frac{1}{16}\right)^{1/2}\right] - \left(\ln\frac{1}{4} - \ln\frac{1}{16}\right) \\
&= 2\left(\frac{1}{2} - \frac{1}{4}\right) - \ln\frac{16}{4} \\
&= \frac{1}{2} - \ln 4.
\end{aligned}$$

□

1.3 Working with Integrals

Theorem 1.3 (Integrals of Even and Odd Functions)**.** Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, i.e., $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, i.e., $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof.

- If f is even, i.e., $f(-x) = f(x)$, then we have

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\lim_{n\to\infty}\sum_{k=1}^{n} f\left(0 + \frac{-ka}{n}\right)\frac{-a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(0 + \frac{ka}{n}\right)\frac{a}{n} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{-ka}{n}\right)\frac{a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} \\
&= 2\left[\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n}\right] \\
&= 2\int_{0}^{a} f(x)\,dx.
\end{aligned}$$

- If f is odd, i.e., $f(-x) = -f(x)$, then we have

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\lim_{n\to\infty}\sum_{k=1}^{n} f\left(0 + \frac{-ka}{n}\right)\frac{-a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(0 + \frac{ka}{n}\right)\frac{a}{n} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{-ka}{n}\right)\frac{a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} -f\left(\frac{ka}{n}\right)\frac{a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} \\
&= -\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} + \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\frac{a}{n} \\
&= 0.
\end{aligned}$$

□

Example 1.2 (Integrating symmetric functions)**.** Evaluate the following integrals using symmetric arguments.

(a) $\displaystyle\int_{-2}^{2}(x^4 - 3x^3)\,dx$.

(b) $\displaystyle\int_{-\pi/2}^{\pi/2}(\cos x - 4\sin^3 x)\,dx$.

SOLUTION.

(a)

$$\begin{aligned}\int_{-2}^{2}(x^4 - 3x^3)\,dx &= \int_{-2}^{2} x^4\,dx - 3\int_{-2}^{2} x^3\,dx \\ &= 2\int_{0}^{2} x^4\,dx - 0 \\ &= 2\frac{x^5}{5}\bigg|_0^2 \\ &= \frac{2\times 2^5}{5} \\ &= \frac{64}{5}.\end{aligned}$$

(b)

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}(\cos x - 4\sin^3 x)\,dx &= \int_{-\pi/2}^{\pi/2}\cos x\,dx - 4\int_{-\pi/2}^{\pi/2}\sin^3 x\,dx \\ &= 2\int_{0}^{\pi/2}\cos x\,dx - 0 \\ &= 2\sin x\bigg|_0^{\pi/2} \\ &= 2.\end{aligned}$$

□