MATH 2205 - Calculus II Lecture Notes 01

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1 Integration

1.1 Sigma (Summation) Notation

Sigma (or *summation*) *notation* is used to express sums in compact way.

Definition 1.1 (Sigma Notation)**.** The symbol Σ (*sigma*, the Greek capital S) stands for *sum*. Then index k takes on all integer values from the lower limit $k = 1$ to the upper limit $k = 10$. The expression that immediately follows Σ (the *summand*) is evaluated for each value of k, and the resulting values are summed.

Example 1.1.

(a) $\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15.$

(b) $\sum_{i=-2}^{2} i = (-2) + (-1) + 0 + 1 + 2 = 0.$

(c) $\sum_{j=1}^{3} (j^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) = 2 + 5 + 10 = 17.$

Proposition 1.1. Suppose that $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two sets of real numbers, and suppose that $c \in \mathbb{R}$ is a real number.

- *Constant Multiple Rule.* We can factor multiplicative constants out of sum:

$$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k.$$

- *Addition Rule.* We can also split a sum into two sums:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

Theorem 1.1 (Sums of Powers of Integers)**.** Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

1.2 Approximating Areas under Curves

Definition 1.2 (Regular Partition)**.** Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

of equal length $\Delta x = (b-a)/n$ with $x_0 = a$ and $x_n = b$. The endpoints $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ of the subintervals are called *grid points*, and they create a *regular partition* of the interval $[a, b]$. In general, the kth point is

$$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \ldots, n.$$

Definition 1.3 (Riemann Sum)**.** Suppose f is a function defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

is called a *Riemann sum* for f on $[a, b]$. This sum is called

- a *left Riemann sum* if x_k^* is the left endpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = x_{k-1} = a + (k-1)\Delta x$;
- a *right Riemann sum* if x_k^* is the right endpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = x_k = a + k\Delta x$; and
- a *midpoint Riemann sum* if x_k^* is the midpoint of $[x_{k-1}, x_k]$, i.e., $x_k^* = (x_{k-1} + x_k)/2 = a + (k - 1/2)\Delta x$, for $k = 1, 2, \ldots, n$.

Example 1.2. Let R be the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 8$.

(a) Approximate the area of R using a left Riemann sum with $n = 8$ subintervals.
(b) Approximate the area of R using a right Riemann sum with $n = 8$ subintervals.
(c) Approximate the area of R using a midpoint Riemann sum with $n = 8$ subintervals.

SOLUTION. By the definition, the Riemann sum for the approximation of the area of R is

$$\sum_{k=1}^{8} f(x_k^*)\Delta x, \tag{1.1}$$

where $\Delta x = \dfrac{x_8 - x_0}{8} = \dfrac{8-0}{8} = 1$.

(a) For a left Riemann sum, we pick $x_k^* = x_{k-1} = x_0 + (k-1)\Delta x = 0 + (k-1) \times 1 = k - 1$. Then plug it into (1.1), we have

$$\begin{aligned}
\sum_{k=1}^{8} f(x_k^*)\Delta x &= \sum_{k=1}^{8} (k-1)^2 \times 1 \\
&= \sum_{k=1}^{8} (k^2 - 2k + 1) \qquad [\text{by } (a-b)^2 = a^2 - 2ab + b^2] \\
&= \sum_{k=1}^{8} k^2 + \sum_{k=1}^{8} (-2k) + \sum_{k=1}^{8} 1 \qquad [\text{by Proposition 1.1 Addition Rule}] \\
&= \sum_{k=1}^{8} k^2 + (-2)\sum_{k=1}^{8} k + \sum_{k=1}^{8} 1 \qquad [\text{by Proposition 1.1 Constant Multiple Rule}] \\
&= \frac{8(8+1)(2 \times 8 + 1)}{6} + (-2)\frac{8(8+1)}{2} + 8 \times 1 \qquad [\text{by Theorem 1.1}] \\
&= 204 - 72 + 8 \\
&= 140.
\end{aligned}$$

(b) For a right Riemann sum, we pick $x_k^* = x_k = x_0 + k\Delta x = 0 + k \times 1 = k$. Then we have

$$\begin{aligned}\sum_{k=1}^{8} f(x_k^*)\Delta x &= \sum_{k=1}^{8} k^2 \times 1 \\ &= \sum_{k=1}^{8} k^2 \\ &= \frac{8(8+1)(2\times 8+1)}{6} \qquad \text{[by Theorem 1.1]} \\ &= 204.\end{aligned}$$

(c) For a mid Riemann sum, we pick $x_k^* = (x_{k-1} + x_k)/2 = x_0 + (k - 1/2)\Delta x = 0 + (k - 1/2) = k - 1/2$. Then we have

$$\begin{aligned}\sum_{k=1}^{8} f(x_k^*)\Delta x &= \sum_{k=1}^{8} \left(k - \frac{1}{2}\right)^2 \times 1 \\ &= \sum_{k=1}^{8} \left(k^2 - k + \frac{1}{4}\right) \qquad \text{[by } (a-b)^2 = a^2 - 2ab + b^2\text{]} \\ &= \sum_{k=1}^{8} k^2 - \sum_{k=1}^{8} k + \sum_{k=1}^{8} \frac{1}{4} \qquad \text{[by Proposition 1.1]} \\ &= \frac{8(8+1)(2\times 8+1)}{6} - \frac{8(8+1)}{2} + 8 \times \frac{1}{4} \\ &= 204 - 36 + 2 \\ &= 170.\end{aligned}$$

□

1.3 Definite Integrals

Definition 1.4 (Net Area)**.** Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The *net area* of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

Definition 1.5 (General Riemann Sum)**.** Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$. If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a *general Riemann sum* for f on $[a, b]$.

Definition 1.6 (Definite Integral)**.** A function f defined on $[a,b]$ is *integrable* on $[a,b]$ if

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and is unique over all partitions of $[a,b]$ and all choices of x_k^* on a partition. This limit is the *definite integral* of f from a to b, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

To simplify the calculation, we use equally spaced grid points and right Riemann sums. That is, for each value of n, we let $\Delta x_k = \Delta x = (b-a)/n$ and $x_k^* = a + k\Delta x$, for $k = 1, 2, \ldots, n$. Then $n \to \infty$ and $\Delta \to 0$,

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

Definition 1.7 (Integrable Functions)**.** If f is continuous on $[a,b]$ or bounded on $[a,b]$ with a finite number of discontinuities, then f is integrable on $[a,b]$.

Definition 1.8 (Reversing Limits and Identical Limits of Integration)**.** Suppose f is integrable on $[a,b]$.

(a) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

(b) $\displaystyle\int_a^a f(x)\,dx = 0.$

Proposition 1.2 (Properties of Definite Integrals)**.**

(a) *Integral of a Sum.* Assume f and g are integrable on $[a,b]$, then their sum $f+g$ is integrable on $[a,b]$ and the integral of their sum is the sum of their integrals:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(b) *Constants in Integrals.* If f is integrable on $[a,b]$ and c is a constant, then cf is integrable on $[a,b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

(c) *Integrals over Subintervals.* If the point p is distinct from a and b, then the integral on $[a,b]$ may be split into two integrals,

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx.$$

Example 1.3. Find the value of $\int_0^2 (x^3+1)\,dx$ by evaluating a right Riemann sum and letting $n \to \infty$.

Solution. The interval $[a,b] = [0,2]$ is divided into n subintervals of length $\Delta x = (b-a)/n = (2-0)/n = 2/n$. For the right Riemann sum, we pick $x_k^* = x_k = x_0 + k\Delta x = 2k/n$. Let $f(x)$ be the integrand, i.e., $f(x) = x^3 + 1$. Then

$$
\begin{aligned}
\int_0^2 f(x)\,dx &= \int_0^2 (x^3+1)\,dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(\frac{2k}{n}\right)^3 + 1\right]\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2^3}{n^3}k^3 + 1\right)\frac{2}{n} && \left[\text{by } \left(\frac{ab}{c}\right)^d = \frac{a^d b^d}{c^d}\right] \\
&= \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} \left(\frac{2^3}{n^3}k^3 + 1\right) && [\text{by Proposition 1.1 Constant Multiple Rule}] \\
&= \lim_{n\to\infty} \frac{2}{n}\left(\sum_{k=1}^{n} \frac{2^3}{n^3}k^3 + \sum_{k=1}^{n} 1\right) && [\text{by Proposition 1.1 Addition Rule}] \\
&= \lim_{n\to\infty} \frac{2}{n}\frac{2^3}{n^3}\sum_{k=1}^{n} k^3 + \frac{2}{n}\sum_{k=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{2^4}{n^4}\sum_{k=1}^{n} k^3 + \frac{2}{n}\sum_{k=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{16}{n^4}\frac{n^2(n+1)^2}{4} + \frac{2}{n}n && [\text{by Theorem 1.1}] \\
&= \lim_{n\to\infty} \frac{4(n+1)^2}{n^2} + 2 \\
&= \lim_{n\to\infty} 4\left(\frac{n+1}{n}\right)^2 + 2 \\
&= \lim_{n\to\infty} 4\left(1+\frac{1}{n}\right)^2 + 2 \\
&= 4 + 2 \\
&= 6.
\end{aligned}
$$

□

Example 1.4. Assume that $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10$. Evaluate the following integrals.

(a) $\displaystyle\int_0^7 2f(x)\,dx = 2\int_0^7 f(x)\,dx = 2 \times (-10) = -20.$

(b) $\displaystyle\int_5^7 f(x)\,dx = \int_5^0 f(x)\,dx + \int_0^7 f(x)\,dx = -\int_0^5 f(x)\,dx + (-10) = -3 + (-10) = -13.$

(c) $\displaystyle\int_5^0 f(x)\,dx = \int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -3.$

(d) $\displaystyle\int_7^0 6f(x)\,dx = 6\int_7^0 f(x)\,dx = -6\int_0^7 f(x)\,dx = (-6)\times(-10) = 60.$