Full Name: ______________________ **W-Number:** ________________

MATH 2205: Calculus II – Final Exam Solution

Summer 2019 – Friday, July 05, 2019

Instructions:

- Show all your work and use the space provided on the exam. Correct mathematical notation is required and all partial credit is at discretion of the grader.
- Write neatly and make sure your work is organized.
- Make certain that you have written your Full Name and W-Number in the spaces provided at the top of the exam. Failure to do so may result in a loss of points.
- No aids beyond a scientific, non-graphing calculator are allowed. This means no notes, no cell phones, etc., are permitted during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 10 pages. Please check to see that your copy has all the pages.

For Instructor Use Only

Question	1	2	3	4	5	6	7	8	9	Total
Points	15	20	10	10	10	10	10	15	10	100
Mark										

Formulae you may find useful:

- $\sum_{k=1}^{n} c = cn, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$
- $\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$, where $p \neq -1$.
- $\int x^{-1} \, dx = \ln|x| + C.$
- $\int e^x \, dx = e^x + C.$
- $\int \sin x \, dx = -\cos x + C.$
- $\int \cos x \, dx = \sin x + C.$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C.$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C.$
- $\int \sec x \tan x \, dx = \sec x + C.$
- $\int \sec^2 x \, dx = \tan x + C.$
- $\cos^2\theta = \frac{1+\cos 2\theta}{2}.$
- $\sin^2\theta = \frac{1-\cos 2\theta}{2}.$
- $\sin^2\theta + \cos^2\theta = 1.$
- $\sin 2\theta = 2\sin\theta\cos\theta.$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta.$
- Midpoint Rule: $M(n) = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x.$
- Trapezoid Rule: $T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x.$
- Simpson's Rule: $S(n) = \sum_{k=0}^{n/2-1} \left[f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right]\frac{\Delta x}{3}.$

1. (15 points) Circle TRUE if the statement is true or FALSE if it is not, and justify your choice briefly.

   (a) TRUE or $\boxed{\text{FALSE}}$: The partial fractions decomposition of $\dfrac{1}{(4k+1)(4k+3)}$ is $\dfrac{1}{4k+1} - \dfrac{1}{4k+3}$.
   Explanation:

$$\frac{1}{4k+1} - \frac{1}{4k+3} = \frac{(4k+3)-(4k+1)}{(4k+1)(4k+3)} = \frac{2}{(4k+1)(4k+3)} \neq \frac{1}{(4k+1)(4k+3)}.$$

   (b) $\boxed{\text{TRUE}}$ or FALSE: The interval of convergence of the series $\displaystyle\sum_{k=1}^{\infty} c_k(x-a)^k$ must contain the center $x = a$.
   Explanation:

$$\left.\sum_{k=1}^{\infty} c_k(x-a)^k\right|_{x=a} = \sum_{k=1}^{\infty} c_k(a-a)^k = 0.$$

   Then the power series must converge at the center $x = a$.

   (c) TRUE or $\boxed{\text{FALSE}}$: The Fundamental Theorem of Calculus uses the derivative of the function f to evaluate the definite integral $\displaystyle\int_a^b f(x)\,dx$.
   Explanation: FTOC uses antiderivative.

   (d) $\boxed{\text{TRUE}}$ or FALSE: If $\displaystyle\sum_{k=0}^{\infty}(\cos k \cdot a_k)$ diverges, then $\displaystyle\sum_{k=0}^{\infty}|a_k|$ must diverge.
   Explanation: Note that $\cos k \cdot a_k \leq |\cos k \cdot a_k| = |\cos k||a_k| \leq |a_k|$ because $|\cos k| \leq 1$. Then by Comparison Test, if $\displaystyle\sum_{k=0}^{\infty}(\cos k \cdot a_k)$ diverges, then $\displaystyle\sum_{k=0}^{\infty}|a_k|$ must diverge.

   (e) $\boxed{\text{TRUE}}$ or FALSE: $\dfrac{[2(k+1)]!}{(2k)!} = (2k+1)(2k+2)$.
   Explanation:

$$\frac{[2(k+1)]!}{(2k)!} = \frac{(2k+2)!}{(2k)!} = \frac{(2k)!\,(2k+1)(2k+2)}{(2k)!} = (2k+1)(2k+2).$$

2. (20 points) Evaluate the following sums.

   (a) $\displaystyle\sum_{k=11}^{50}(k+3)(k-3).$

   Solution.

$$\begin{aligned}
\sum_{k=11}^{50}(k+3)(k-3) &= \sum_{k=11}^{50}(k^2-9)\\
&= \sum_{k=1}^{50}(k^2-9) - \sum_{k=1}^{10}(k^2-9)\\
&= \left[\sum_{k=1}^{50}k^2 - \sum_{k=1}^{50}9\right] - \left[\sum_{k=1}^{10}k^2 - \sum_{k=1}^{10}9\right]\\
&= \frac{n(n+1)(2n+1)}{6} - 9n\Big|_{n=10}^{n=50}\\
&= \left(\frac{50\cdot 51\cdot 101}{6} - 9\cdot 50\right) - \left(\frac{10\cdot 11\cdot 21}{6} - 9\cdot 10\right)\\
&= 41280.
\end{aligned}$$

□

   (b) $\displaystyle\sum_{k=0}^{\infty}\frac{3}{(3k+1)(3k+4)}.$

   Solution.

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{3}{(3k+1)(3k+4)}\\
=\ &\sum_{k=0}^{\infty}\frac{(3k+4)-(3k+1)}{(3k+1)(3k+4)}\\
=\ &\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}-\frac{1}{3k+4}\right)\\
=\ &\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{1}{3k+1}-\frac{1}{3k+4}\right)\\
=\ &\lim_{n\to\infty} 1-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{11}+\cdots+\frac{1}{3n-2}-\frac{1}{3n+1}+\frac{1}{3n+1}-\frac{1}{3n+4}\\
=\ &\lim_{n\to\infty} 1-\frac{1}{3n+4}\\
=\ &1.
\end{aligned}$$

□

(c) $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\cdots$.

SOLUTION. Observe that the series is a geometric series with the ratio $r=\frac{a_{k+1}}{a_k}=\frac{8/81}{4/27}=\frac{4/27}{2/9}=\cdots=\frac{2}{3}$, $a=\frac{1}{3}$, $m=0$. Then

$$\begin{aligned}
\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\cdots &= \sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^k && [a=\frac{1}{3}, r=\frac{2}{3}, m=0]\\
&= \frac{\frac{1}{3}\left(\frac{2}{3}\right)^0}{1-\frac{2}{3}} && [\sum_{k=m} ar^k=\frac{ar^m}{1-r}]\\
&= \frac{1}{3}\frac{3}{1}\\
&= 1.
\end{aligned}$$

□

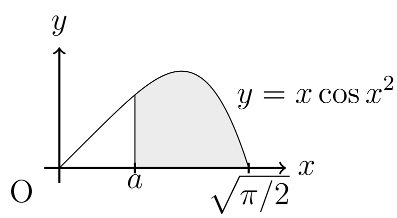
(d) $\sum_{k=0}^{\infty}\left(e^{-2k-1}+\frac{1}{\pi^{3k+1}}\right)$.

SOLUTION.

$$\begin{aligned}
\sum_{k=0}^{\infty}\left(e^{-2k-1}+\frac{1}{\pi^{3k+1}}\right) &= \sum_{k=0}^{\infty}e^{-2k-1}+\sum_{k=0}^{\infty}\frac{1}{\pi^{3k+1}}\\
&= \sum_{k=0}^{\infty}e^{-1}\cdot(e^{-2})^k+\sum_{k=0}^{\infty}\pi^{-1}\left(\frac{1}{\pi^3}\right)^k\\
&= \frac{e^{-1}\cdot(e^{-2})^0}{1-e^{-2}}+\frac{\pi^{-1}(\pi^{-3})^0}{1-\pi^{-3}} && [\sum_{k=m}^{\infty} ar^k=\frac{ar^m}{1-r}]\\
&= \frac{e^{-1}}{1-e^{-2}}+\frac{\pi^{-1}}{1-\pi^{-3}}\\
&= \frac{e}{e^2-1}+\frac{\pi^2}{\pi^3-1}.
\end{aligned}$$

□

3. (10 points) Determine the value of the positive parameter a so that the area of the shaded region in the picture is equal to $\frac{1}{2}$.



Solution. Let $f(x) = x\cos x^2$. The area of the shaded region is

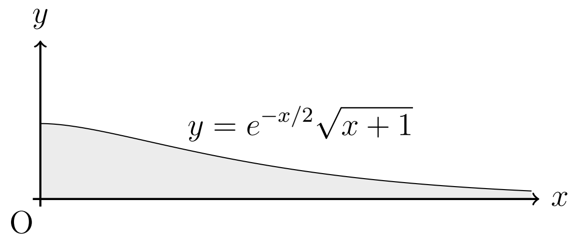
$$\begin{aligned}
A &= \int_a^b f(x)\,dx \\
&= \int_a^{\sqrt{\pi/2}} x\cos x^2\,dx \\
&= \frac{1}{2}\int_a^{\sqrt{\pi/2}} \cos x^2 (2x)\,dx \\
&= \frac{1}{2}\int_{u(a)=a^2}^{u(\sqrt{\pi/2})=\pi/2} \cos u\,du \qquad [u = x^2, du = 2x\,dx] \\
&= \frac{1}{2}\sin u\Big|_{a^2}^{\pi/2} \\
&= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin a^2\right) \\
&= \frac{1}{2} - \frac{1}{2}\sin a^2.
\end{aligned}$$

Given that the area of the shaded region $A = \frac{1}{2}$, we have

$$\frac{1}{2} - \frac{1}{2}\sin a^2 = \frac{1}{2} \implies \sin a^2 = 0 \implies a^2 = 0 \implies a = 0.$$

$\square$

4. (10 points) Consider the infinitely long shaded region R indicated in the picture. Determine the volume of the solid of the revolution obtained when R is revolved about the x-axis.



Solution. We can use the Disk Method to calculate the volume of solid of the revolution. Let $f(x) = e^{-x/2}\sqrt{x+1} = e^{-x/2}(x+1)^{1/2}$, then

$$\begin{aligned}
V &= \int_a^b \pi f(x)^2\,dx \\
&= \int_0^\infty \pi[e^{-x/2}(x+1)^{1/2}]^2\,dx \\
&= \pi \lim_{b\to\infty}\int_0^b e^{-x}(x+1)\,dx \\
&= \pi\left(\lim_{b\to\infty}\int_0^b e^{-x}x\,dx + \lim_{b\to\infty}\int_0^b e^{-x}\,dx\right) \\
&= \pi\left(\lim_{b\to\infty}(-e^{-x})x\Big|_0^b - \int_0^b(-e^{-x})\,dx + \lim_{b\to\infty}(-e^{-x})\Big|_0^b\right) \\
&= \pi\left[\lim_{b\to\infty} -e^{-b}b + e^0\cdot 0 + \int_0^b e^{-x}\,dx + \lim_{b\to\infty}\left(-e^{-b}+e^0\right)\right] \\
&= \pi\left(\lim_{b\to\infty} -e^{-x}\Big|_0^b + 1\right) \\
&= \pi\left(\lim_{b\to\infty} -e^{-b} + e^0 + 1\right) \\
&= 2\pi.
\end{aligned}$$

$\square$

5. (10 points) Find the interval of convergence and radius of convergence for the power series given by

$$\sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{k^{4/5}}.$$

SOLUTION. Since the Ratio/Root Test requires positive terms. Here we test the absolute convergence of the given series using Ratio Test. We can identify that $a_k = \dfrac{(-1)^k (x-1)^k}{k^{4/5}}$. Then

$$\begin{aligned}
r &= \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \\
&= \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(x-1)^{k+1}/(k+1)^{4/5}}{(-1)^k (x-1)^k / k^{4/5}} \right| \\
&= \lim_{k\to\infty} \left| \frac{(-1)^k \cdot (-1)(x-1)^k \cdot (x-1)}{(k+1)^{4/5}} \frac{k^{4/5}}{(-1)^k (x-1)^k} \right| \\
&= \lim_{k\to\infty} \left| \frac{(-1)(x-1)k^{4/5}}{(k+1)^{4/5}} \right| \\
&= \lim_{k\to\infty} |x-1| \left(\frac{k}{k+1} \right)^{4/5} \\
&= |x-1| \lim_{k\to\infty} \left(\frac{1}{1+1/k} \right)^{4/5} \\
&= |x-1|.
\end{aligned}$$

The Ratio Test requires $r = |x-1| < 1$, then we have

$$|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2.$$

However, the Ratio (Root) Test is inconlusive when r (ρ) is 1. Therefore, we need to check the endpoints of the interval of convergence $x = 0$ and $x = 2$. When $x = 0$, the series becomes

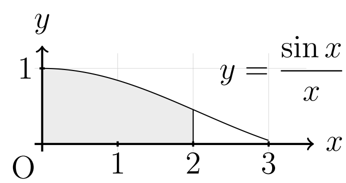
$$\sum_{k=1}^{\infty} \frac{(-1)^k (0-1)^k}{k^{4/5}} = \sum_{k=1}^{\infty} \frac{(-1)^k (-1)^k}{k^{4/5}} = \sum_{k=1}^{\infty} \frac{[(-1)^2]^k}{k^{4/5}} = \sum_{k=1}^{\infty} \frac{1}{k^{4/5}},$$

which is a p-series with $p = \dfrac{4}{5} < 1$. Hence when $x = 0$, the series diverges. When $x = 2$, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^k (2-1)^k}{k^{4/5}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{4/5}},$$

which is an alternating series. We can perform the Alternating Series Test, noting $\dfrac{1}{k^{4/5}}$ is decreasing as k increases, and $\lim_{k\to\infty} \frac{1}{k^{4/5}} = 0$. Therefore, the series converges when $x = 2$. Thus, the interval of convergence is $(0, 2]$, i.e., $0 < x \le 2$. Therefore, the radius of convergence is $R = 1$. □

6. (10 points) Use MacLaurin Series to approximate the net area between the function $y = \dfrac{\sin x}{x}$ and the x-axis from $x = 0$ to $x = 2$ with an error no greater than $10^{-4} = 0.0001$. Be sure to justify that your error satisfies the given bound.



Solution. Let $f(x) = \dfrac{\sin x}{x}$, then the area of the region is

$$A = \int_a^b f(x)\,dx = \int_0^2 \frac{\sin x}{x}\,dx = \int_0^2 x^{-1}\sin x\,dx.$$

Recall that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Then the integral becomes

$$\begin{aligned}
A &= \int_0^2 x^{-1}\sin x\,dx \\
&= \int_0^2 x^{-1}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\,dx \\
&= \int_0^2 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}\,dx \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\int_0^2 x^{2k}\,dx \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{x^{2k+1}}{2k+1}\Bigg|_{x=0}^{x=2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{2^{2k+1}}{2k+1}
\end{aligned}$$

Noting that the second term is an alternating series, then we need to find n such that the partial sum $S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\frac{2^{2k+1}}{2k+1}$ has error less than 10^{-4}, then

$$|R_n| < a_{n+1} = \frac{1}{[2(n+1)+1]!}\frac{2^{2(n+1)+1}}{2(n+1)+1} = \frac{2^{2n+3}}{(2n+3)!\,(2n+3)} < 10^{-4}.$$

Solve for n, we have $n > 3$, then we pick $n = 4$, which gives $A \approx S_4 = \sum_{k=0}^{4} \frac{(-1)^k}{(2k+1)!}\frac{2^{2k+1}}{2k+1} \approx 1.605417542$. □

7. (10 points) Find MacLaurin Series for $\ln(1+2x)$ and give its interval of convergence.

Solution. Observe that

$$\ln(1+2x) = \int \frac{2}{1+2x}\,dx.$$

Recall that $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x|< 1$, then

$$\frac{2}{1+2x} = 2\frac{1}{1-(-2x)} = 2\sum_{k=0}^{\infty}(-2x)^k = 2\sum_{k=0}^{\infty}(-2)^k x^k \text{ for } |-2x|< 1 \implies |x|< \frac{1}{2}.$$

Then we can substitute the MacLaurin series into the integrals

$$\begin{aligned}
\ln(1+2x) &= \int \frac{2}{1+2x}\,dx \\
&= \int 2\sum_{k=0}^{\infty}(-1)^k(2x)^k\,dx \\
&= 2\sum_{k=0}^{\infty}(-1)^k 2^k \int x^k\,dx \\
&= 2\sum_{k=0}^{\infty}(-1)^k 2^k \frac{x^{k+1}}{k+1} + C \\
&= \sum_{k=0}^{\infty}(-1)^k 2^{k+1} \frac{x^{k+1}}{k+1} + C \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{(2x)^{k+1}}{k+1} + C.
\end{aligned}$$

Note that when $x = 0$, $\ln(1+2x)\big|_{x=0} = \ln 1 = 0$. Then, we have $C = 0$. Hence,

$$\ln(1+2x) = \sum_{k=0}^{\infty}(-1)^k \frac{(2x)^{k+1}}{k+1} \text{ for } |x|< \frac{1}{2}.$$

Note that when $x = -\frac{1}{2}$, the series becomes

$$\sum_{k=0}^{\infty}(-1)^k \frac{(-1)^{k+1}}{k+1} = \sum_{k=0}^{\infty}\frac{(-1)^{2k+1}}{k+1} = -\sum_{k=0}^{\infty}\frac{1}{k+1},$$

which is a harmonic series, hence, it diverges at $x = -\frac{1}{2}$. When $x = \frac{1}{2}$, the series becomes

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1},$$

which is an alternating harmonic series, thus the series converges at $x = \frac{1}{2}$. Therefore, the interval of convergence is $(-\dfrac{1}{2}, \dfrac{1}{2}]$, i.e, $-\dfrac{1}{2} < x \leq \dfrac{1}{2}$. □

8. (15 points) Determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k^4+10k}{2k^4+50}\right)^k$.

SOLUTION. Root Test. Identify that $a_k = \left(\dfrac{k^4+10k}{2k^4+50}\right)^k$. Then

$$\begin{aligned}
\rho = \lim_{k\to\infty}\sqrt[k]{a_k} &= \lim_{k\to\infty}(a_k)^{1/k} \\
&= \lim_{k\to\infty}\left[\left(\frac{k^4+10k}{2k^4+50}\right)^k\right]^{1/k} \\
&= \lim_{k\to\infty}\frac{k^4+10k}{2k^4+50} \\
&= \lim_{k\to\infty}\frac{1+10k^{-3}}{2+50k^{-4}} \\
&= \frac{1}{2}.
\end{aligned}$$

By Root Test, $\rho = \dfrac{1}{2} < 1$, then the given series converges. □

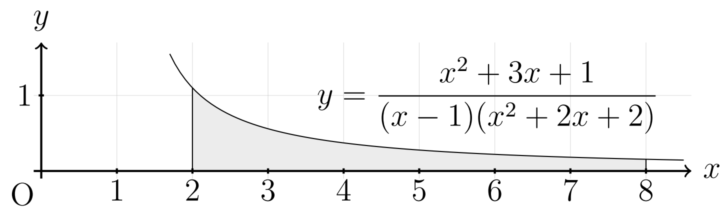
(b) $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{k!}$.

SOLUTION. Ratio Test. Note that $a_k = \dfrac{k^k}{k!}, a_{k+1} = \dfrac{(k+1)^{k+1}}{(k+1)!}$,

$$\begin{aligned}
r = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} &= \lim_{k\to\infty}\frac{(k+1)^{k+1}/(k+1)!}{k^k/k!} \\
&= \lim_{k\to\infty}\frac{(k+1)^k\cdot(k+1)}{k!\cdot(k+1)}\frac{k!}{k^k} \\
&= \lim_{k\to\infty}\frac{(k+1)^k}{k^k} \\
&= \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k \\
&= \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k \\
&= e.
\end{aligned}$$

By Ratio Test, $r = e > 1$, the given series diverges. □

9. (Bonus, 10 points) Compute the area of region R bounded by $y = \dfrac{x^2+3x+1}{(x-1)(x^2+2x+2)}$, x-axis, $x = 2$, $x = 8$.



SOLUTION. The area of the region R is

$$A = \int_a^b f(x)\,dx = \int_2^8 \frac{x^2+3x+1}{(x-1)(x^2+2x+2)}\,dx.$$

Then we apply the partial fraction decomposition to the integrand,

$$\frac{x^2+3x+1}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}.$$

Multiplying both sides by $(x-1)(x^2+2x+2)$ gives

$$x^2+3x+1 = A(x^2+2x+2)+(Bx+C)(x-1) = (A+B)x^2+(2A-B+C)x+(2A-C).$$

Equating like powers of x, we have the following linear equations,

$$\begin{cases} A+B=1 \\ 2A-B+C=3 \\ 2A-C=1 \end{cases} \implies \begin{cases} A=1 \\ B=0 \\ C=1 \end{cases}$$

Then definite integral becomes

$$\begin{aligned} A &= \int_2^8 \frac{x^2+3x+1}{(x-1)(x^2+2x+2)}\,dx \\ &= \int_2^8 \frac{1}{x-1} + \frac{1}{x^2+2x+2}\,dx \\ &= \int_2^8 \frac{1}{x-1}\,dx + \frac{1}{(x^2+2x+1)+1}\,dx \\ &= \ln(x-1)\Big|_2^8 + \int_2^8 \frac{1}{1+(x+1)^2}\,dx \\ &= \ln 7 - \ln 1 + \arctan(x+1)\Bigg|_2^8 \\ &= \ln 7 + \arctan 9 - \arctan 3. \end{aligned}$$

□