MATH 5290 - Operator Algebras & K-Theory Lecture Notes 2

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1 About K-theory

K-thoery was developed by Atiyah and Hirzebruch in the 1960s based on work of Grothendieck in algebraic geometry. It was introduced as a tool in C^*-algebra theory in the early 1970 through some specific applications described below. Very briefly, K-theory (for C^*-algebra) is a pair of functors, called K_0 and K_1 that to each C^*-algebra A associate two Abelian groups $K_0(A)$ and $K_1(A)$. The group $K_0(A)$ is given an ordering that (in special cases) makes it an ordered Abelian group. There are powerful machines, some of which are described in this book, making it possible to calculate the K-theory of a great many C^*-algebras. K-theory contains much information about the individual C^*-algebra – one can learn about the structure of a given C^*-algebras from each other by distinguishing their K-theories. For certain classes of C^*-algebras, K-theory is actually a complete invariant. K-theory is also a natural home for index theory.

2 Elementary Spectral Theory

2.1 Banach Algebras

Definition 2.1 (Algebra)**.** An *algebra* is a vector space A together with a bilinear map

$$A^2 \to A, \quad (a, b) \mapsto ab,$$

such that

$$a(bc) = (ab)c \quad (a, b, c \in A).$$

Definition 2.2 (Subalgebra)**.** A *subalgebra* of A is a vector subspace B such that $b, b' \in B \implies bb' \in B$.

3 C^*-Algebra Theory

3.1 C^*-algebras and *-homomorphisms

Definition 3.1. A C^*-algebra A is an algebra over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*, a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for every $a, b \in A$.

The axioms for a C^*-algebra A above imply that involution is isometric, i.e., $\|a\| = \|a^*\|$ for every a in A.

Definition 3.2. A C^*-algebra A is called *unital* if it has a multiplicative identity, which will be denoted by 1 or 1_A. A *-homomorphism $\varphi : A \to B$ between C^*-algebras A and B is a linear and multiplicative map which satisfies $\varphi(a^*) = \varphi(a)^*$ for all a in A. If A and B are unital and $\varphi(1_A) = 1_B$, then φ is called *unital* (or *unit preserving*). A C^*-algebra is said to be *separable* if it contains a countable dense subset.

Definition 3.3 (Sub-C^*-algebras and sub-*-algebras)**.** A non-empty subset B of a C^*-algebra A is called a *sub-*-algebra* of A if it is a *-algebra with the operations given on A, that is, if it is closed under the algebraic operations: A *sub-C^*-algebra* of A is a non-empty subset of A which is a C^*-algebra with respect to the operations given on A. Hence, a non-empty subset B of a C^*-algebra A is a sub-C^*-algebra if and only if it is norm-closed and closed under the four algebraic operations listed above.

addition	$A \times A \to A,$	$(a, b) \mapsto a + b$
multiplication	$A \times A \to A,$	$(a, b) \mapsto ab$
adjoint	$A \to A,$	$a \mapsto a^*$
scalar multiplication	$\mathbb{C} \times A \to A,$	$(\alpha, a) \mapsto \alpha a$

Example 3.1. Let A be a C^*-algebra, and let F be a subset of A. The sub-C^*-algebra of A generated by F, denoted by $C^*(F)$, is the smallest sub-C^*-algebra of A that contains F. In other words, $C^*(F)$ is the intersection of all sub-C^*-algebras of A that contain F. The C^*-algebra $C^*(F)$ can be concretely described as follows. For each natural number n put

$$W_n = \{x_1 x_2 \cdots x_n : x_j \in F \cup F^*\},$$

where $F^* = \{x^* : x \in F\}$, and put $W = \bigcup_{n=1}^{\infty} W_n$. The set W is the set of all *words* in $F \cup F^*$, and W_n is the set of words of length n. Using that $W = W^*$ and that W is closed under multiplication, we see that the linear span of W is a sub-*-algebra of A. It follows that

$$C^*(F) = \overline{\operatorname{span} W}.$$

We write $C^*(a_1, a_2, \ldots, a_n)$ instead of $C^*(\{a_1, a_2, \ldots, a_n\})$, when $a_1, a_2, \ldots, a_n$ are elements in A.

Theorem 3.1 (Gelfand-Naimark)**.** For each C^*-algebra A there exists a Hilbert space H and an isometric *-homomorphism φ from A into $B(H)$, the algebra of all bounded linear operators on H. In other words, every C^*-algebra is isomorphic to a sub-C^*-algebra of $B(H)$. If A is separable, then H can be chosen to be a separable Hilbert space.

Definition 3.4. Every ideal is automatically self-adjoint, and thereby a sub-C^*-algebra. Assume that I is an ideal in a C^*-algebra A. The quotient of A by I is

$$A/I = \{a + I : a \in A\}, \|a + I\| = \inf\{\|a + x\| : x \in I\}, \pi(a) = a + I.$$

In this way A/I becomes a C^*-algebra, $\pi : A \to A/I$ is a *-homomorphism, called the *quotient mapping*, and $I = \operatorname{Ker}(\pi)$. Let $\phi : A \to B$ be a *-homomorphism. Then, automatically, $\|\varphi(a)\| \leq \|a\|$ for all a in A, and ϕ is injective if and only if φ is isometric. The kernel, $\operatorname{Ker}(\varphi)$, of φ is an ideal in A, and the image, $\operatorname{Im}(\varphi) = \varphi(A)$, of φ is a sub-C^*-algebra of B. By the first isomorphism theorem there is one (and only one) *-homomorphism $\varphi_0 : A/\operatorname{Ker}(\phi) \to B$ such that the diagram commutes, i.e., $\varphi_0 \circ \pi = \varphi$. Moreover, φ_0 is injective. A C^*-algebra A is called *simple* if the only ideals in A are the two trivial ideals 0 an A.

Definition 3.5. A (finite or infinite) sequence of C^*-algebras and *-homomorphisms

$$\cdots \to A_n \stackrel{\varphi_n}{\to} A_{n+1} \stackrel{\varphi_{n+1}}{\to} A_{n+2} \to \cdots$$

is said to be *exact* if $\operatorname{Im}(\varphi_n) = \operatorname{Ker}(\varphi_{n+1})$ for all n. An exact sequence of the form

$$0 \to I \stackrel{\varphi}{\to} A \stackrel{\psi}{\to} B \to 0$$

is called *short exact*.