Toeplitz C^*-Algebra: A Survey Paper

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Abstract

This survey paper looks into the application of C^*-algebra to numerical analysis, more specifically, finite sections method for Toeplitz operators. Follows that we can study the invertibility problem in the chosen C^*-algebra (Toeplitz C^*-algebra) to get the stability of the finite section method.

1 Introduction

The goal of functional analysis is to solve equations with infinitely many variables, and that of linear algebra to solve equation in finitely many variables. Numerical analysis builds a bridge between these fields. The subject of numerical analysis – as we understand it – is the investigation and theoretical foundation of approximation methods for operator equations. The finite section method is the most natural and most important approximation method for bounded linear opeartors on a separable Hilbert space [1]. Also, the finite section method for Toeplitz operators with piecewise continuous generating function. Based on the stable convergence of certain Toeplitz matrices, A. Böttcher and S. Roch proved the Fisher-Hartwig conjecture. Due to its rich and beautiful structure, the finite section method for Toeplitz operators became a standard model in the functional-analytic theory of approximation methods [2]. In 1980's, Silbermann discovered a new way of translating the problem of the statbility of the finite section method of Toeplitz operators into an invertibility problem in Banach algebras. Then replace the Banach algebra by C^*-algebras, which enjoy a lot of nice properties that are not shared by general Banach algebras [3]. It is introduced that a general approach of studying invertibility problems in algebras the elements of which are sequences of operators. The proposed approach allows to relate properties of the approximation sequence:

- Stable convergence
- Limiting sets of spectra
- Moore-Penrose invertibility
- Asymptotic behaviour of the condition numbers
- The symbol of the sequence

with corresponding properties of a certain function, the symbol of the sequence. On tops of this, the approach can be applied to finite section method for Toeplitz operators, spline projection methods for singular integral equations with piecewise continuous coefficients and spline projection methods for Mellin operators [2]. This paper mainly covers the first part of [2].

2 Basic definitions

2.1 Approximation method

Definition 2.1 (Approximation method)**.** Let H be a Hilbert space and $L(H)$ be the C^*-algebra of all linear and bounded operators on H, and suppose we are given an operator A in $L(H)$ and a sequence (A_n) of operators $A_n \in L(H)$ tending to A in the strong operator topology of H (i.e. $\|A_n x - Ax\| \to 0$ for all $x \in H$). For instance one can have in mind an operator equation

$$Ax = y \quad (x, y \in H), \tag{1}$$

which is tried to be solved by a certain ”reasonable” approximation method

$$A_n x_n = y \quad (x_n, y \in H). \tag{2}$$

Definition 2.2 (Applicability)**.** The approximation method (A_n) for A is *applicable* if there exists a number n_0 such that the equation (2) possess unique solutions x_n for every $n \geq n_0$ and every right hand side $y \in H$ and if these solutions converge in the norm of H to a solution of (1).

To study the correspondence between (A_n) and A by embedding the sequences we are interested in into a suitably chosen C^*-algebra which owns a special structure. To motivate this structure, we start with examining a typical and stimulating example: the approximation of Toeplitz operators by their finite sections.

3 Finite Sections of Toeplitz Operators

3.1 Toeplitz Operator

Definition 3.1. Let l^2 denote the Hilbert space of all sequences $(x_n)_{n\geq 0}$ of complex numbers with inner product $\langle (x_k), (y_k) \rangle = \sum_{k=0}^{\infty} x_k \overline{y_k}$ and corresponding norm $\|(x_k)\| = \langle (x_k), (x_k) \rangle^{1/2}$.
Given a function $a \in L^\infty(\mathbb{T})$, we let

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{iz}) e^{-inz} dz$$

be its nth Fourier coefficient. The *Toeplitz operator* $T(a) : l^2 \to l^2$ generated by a is defined by

$$T(a)(x_k) = (y_k) \text{ with } y_k = \sum_{j=0}^{\infty} a_{k-j} x_j.$$

This operator is bounded, and its norm is equal to $\sup_{t \in \mathbb{T}} |a(t)|$.

3.1.1 Finite Section of the Toeplitz Operator

The matrices $T_n(a) = (a_{i-j})_{i,j=0}^{n-1}$ are referred to as *Toeplitz matrices*. Introducing operators

$$P_n : l^2 \to l^2, (x_k) \mapsto (x_0, \ldots, x_{n-1}, 0, 0, \ldots),$$

we can identify the Toeplitz matrix $T_n(a)$ (acting on $\mathbb{C}^n$) with the *finite section* $P_n T(a) P_n$ of the Toeplitz operator $T(a)$ (acting on l^2).

3.1.2 Applicability of Finite Section Method

We say that the *finite section method applies to the operator* $T(a)$ if the equations

$$P_n T(a) T_n x^{(n)} = P_n y$$

are uniquely solvable for all $n \geq n_0$ and for all right hand sides $y \in l^2$, and if their solutions $x^{(n)} \in \text{Im}(P_n)$ converges in the l^2-norm to a solution of the equation

$$T(a)x = y.$$

Theorem 3.1. Let a be a piecewise continuous function (i.e. a function possessing one-sided limits at each point of the unit circle $\mathbb{T}$). Then the finite section method applies to $T(a)$ if and only if both operators $T(a)$ and $T(\tilde{a})$ with $\tilde{a}(t) = a(1/t)$ are invertible.

4 C^*-Algebra Techniques

4.1 C^*-Algebra of Sequences of Operators

4.1.1 Reformulation

It is not hard to see that the finite section method applies if and only if the sequence $(T_n(a))$ is *stable*, i.e. if and only if the operators $T_n(a) : \operatorname{Im} P_n \to \operatorname{Im}(P_n)$ are invertible for all sufficiently large n, say $n \geq n_0$, and if the norms of their inverses are uniformly bounded.
The advantage of this reformulation is that stability of a sequence can be translated into an invertibility problem in a suitably chosen C^*-algebra.

4.1.2 C^*-Algebra $\mathcal{F}$

Let $\mathcal{F}$ stand for the section of all sequences (A_n), where $A_n : \operatorname{Im} P_n \to \operatorname{Im} P_n$. Defining operations by

$$(A_n) + (B_n) = (A_n + B_n) \text{ and } (A_n)(B_n) = (A_n B_n),$$

an involution by

$$(A_n)^* = (A_n^*),$$

and a norm by

$$\|(A_n)\| = \sup_n \|A_n\|,$$

one can make $\mathcal{F}$ to become a C^*-algebra. Clearly, the sequence (P_n) is the identity element in this algebra, and it is also easy to check that a sequence $(A_n) \in \mathcal{F}$ is invertible in $\mathcal{F}$ if and only if *all* matrices A_n are invertible and if $\sup_{n\geq 0} \|A_n^{-1}\| < \infty$.

4.1.3 Quotient Algebra $\mathcal{F}/\mathcal{G}$

This is not yet stability, but there is a simple trick to manage this point. Namely, the set $\mathcal{G}$ of all sequences $(G_n) \in \mathcal{F}$ with $\|G_n\| \to 0$ as $n \to \infty$ forms a closed two-sided ideal of the algebra $\mathcal{F}$, and a little thought reveals that the coset $(A_n) + \mathcal{G}$ is invertible in the *quotient algebra* $\mathcal{F}/\mathcal{G}$ if and only if the matrices A_n are invertible beginning with a subscript n_0, and if $\sup_{n\geq n_0} \|A_n^{-1}\| < \infty$, which exactly means stability.

4.2 Toeplitz C^*-Algebra

4.2.1 Toeplitz C^*-Algebra $\mathcal{A}$

For our purposes it is more convenient to work in a smaller algebra than $\mathcal{F}/\mathcal{G}$. Let $\mathcal{A}$ denote the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences $(P_n T(a) P_n)$ with a running through the piecewise continuous functions. One can show that $\mathcal{G} \subset \mathcal{A}$, hence, one can form the quotient algebra $\mathcal{A}/\mathcal{G}$, and this algebra can be viewed as a *-subalgebra of $\mathcal{F}/\mathcal{G}$.

4.2.2 Invertibility of $(A_n) + \mathcal{G}$

One might ask whether invertibility of a coset $(A_n) + \mathcal{G}$ in $\mathcal{F}/\mathcal{G}$ respective in $\mathcal{A}/\mathcal{G}$ correspond to the same "stability". But fortunately, we are dealing with C^*-algebra which implies that, if for a sequence (A_n) the coset $(A_n) + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$, then it is also invertible in $\mathcal{A}/\mathcal{G}$, thus, no problems arise when working in $\mathcal{A}/\mathcal{G}$ rather than in $\mathcal{F}/\mathcal{G}$.

4.2.3 Widom's Formula

We need a further family of operators:

$$W_n : l^2 \to l^2, \quad W_n(x_k) = (x_{n-1}, \cdots, x_1, x_0, \ldots).$$

Obviously, $W_n^2 = P_n$ and $W_n P_n = P_n W_n = W_n$. H. Widom established the formula relating to the finite sections of Toeplitz operator the generating function of which is a product ab of two functions with the product of the finite sections of the Toeplitz operators with generating functions a and b: If $a, b \in L^\infty(\mathbb{T})$, then

$$T_n(ab) = T_n(a)T_n(b) + P_n H(a)H(\tilde{b})P_n + W_n H(\tilde{a})H(b)W_n,$$

where $W(a) : l^2 \to l^2$ refers to the Hankel operator

$$H(a)(x_k) = (y_k) \text{ with } y_k = \sum_{j=0}^{\infty} a_{k+j+1} x_j.$$

4.2.4 A Closed Two-Sided Ideal of $\mathcal{A}$

Widom's formula yields that

$$T_n(a)T_n(b) - T_n(b)T_n(a) = P_n(H(a)H(\tilde{b}) - H(b)H(\tilde{a}))P_n + W_n(H(\tilde{a})H(b) - H(\tilde{a})H(a))W_n,$$

and since the operators $H(a)H(\tilde{b}) - H(b)H(\tilde{a})$ and $H(\tilde{a})H(b) - H(\tilde{b})H(a)$ are compact whenever a and b are piecewise continuous, we see that any two sequences $(T_n(a))$ and $(T_n(b))$ commute modulo sequences of the form $(P_n K_0 P_n + W_n K_1 W_n)$ with compact operators K_0 and K_1.
Pushing forward this observation one can even show that the set $\mathcal{J}$ of all sequences (A_n) with

$$A_n = P_n K_0 P_n + W_n K_1 W_n + G_n,$$

with K_0, K_1 compact and $(G_n) \in \mathcal{G}$ is contained in $\mathcal{A}$, forms a closed two sided ideal of $\mathcal{A}$, and that this ideal is nothing else than the commutator ideal of the algebra $\mathcal{A}$.
The commutativity of the quotient algebra $\mathcal{A}/\mathcal{J}$ gives rise to the hope that it could be tackled by means of Gelfand's local spectral theory.

Theorem 4.1. A sequence $(A_n) \in \mathcal{A}$ is stable if and only if both operators $W_0(A_n)$ and $W_1(A_n)$ are invertible.

Theorem 4.2. The algebra $\mathcal{A}/\mathcal{G}$ is isometrically isomorphic to the smallest closed subalgebra of $L(l^2) \times L(l^2)$ spanned by all pairs $(T(a), T(\tilde{a}))$ with a being piecewise continuous.

References

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